Extensions of Hilbert's Tenth Problem

organized by

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Workshop Summary

Report AIM workshop on Hilbert's Tenth Problem

Organisation of the workshop

During the workshop the following activities took place: lectures, problem sessions and research/discussion sessions.

Lectures

Two types of lectures were programmed into the schedule. First there were a number of introductory lectures which were were meant to introduce non-specialist into the area of Hilbert's Tenth Problem. Since quite a few of the participants came from different backgrounds this was an ideal way to get a feeling of the problems which come up in this area of mathematics. The second type of lectures dealt with more specific subjects and were slightly more oriented towards open problems (e.g. lectures by Pheidas on Hilbert's Tenth Problem (H10) for rings of analytic functions, by Eisenträger on H10 for function fields, by Jarden on decidability questions in large rings of integers, Rubin on ranks of elliptic curves over number fields, by Pop on the elementary equivalence versus isomorphism problem, by Cornelissen on the complexity of undecidable formuals over the rational numbers, by Demeyer on diophantine sets over polynomial rings, Rojas on computational complexity of diophantine equations, ...).

Problem Sessions

During the week there were two problem sessions. The aim of the problem sessions was to let people present various problems which they encountered during their own research in the hope that solutions or hints of solutions might be proposed by the other participants. These problem sessions took place on the first and second afternoon of the workshop and each ran for about two hours and a half. Having these problem sessions at the beginning of the workshop had the advantage that further discussion about these problems could take place during the rest of the week.

Research/Discussion sessions

Three afternoons were exclusively reserved for research/discussion sessions in which the participants could propose to have a discussion session on a topic of their choice and other participants could join one of the various groups. In practice this meant that each afternoon we had four or five groups working in parallel on different questions. The following topics were discussed: the elementary equivalence/isomorphism problem for finitely generated fields, decidability questions in large rings of integers, ranks of elliptic curves and elliptic divisibility sequences, H10 for function fields in positive characteristic, the 1st-order theory and the existential theory of algebraic function fields in positive characteristic, H10 and complexity theory. Some of these discussion groups ran for several afternoons. Some encouraging results were obtained in these discussion groups and these were presented on Friday afternoon.

Results

The following (partial) results were announced on Friday afternoon as a result of the discussion sessions during the week: *Large rings of integers.*

(presented by Jarden)

Hypothesis Let K be an extension of $\mathbf{Q}$ contained in the field of totally real algebraic numbers $\mathbf{Q}^{\mathrm{tr}}$. The theory of the ring $\mathcal{O}_K$ of integers in K has an undecidable theory.

To answer this question, one first need to know about the structure of these fields. The first result was produced at the AIM workshop:

Theorem Let K be a subfield of $\mathbf{Q}^{\mathrm{tr}}$, then the extension $\mathbf{Q}^{\mathrm{tr}}/K$ is infinte.

Some progress towards proving the following conjecture was made at the workshop:

Conjecture For almost all $\sigma \in \mathrm{Gal}(\mathbf{Q}^{\mathrm{tr}}/\mathbf{Q})$, the ring $\mathbf{Z}^{\mathrm{tr}}(\sigma)$ (i.e. the subring of the ring of integers $\mathbf{Z}^{\mathrm{tr}}$ of $\mathbf{Q}^{\mathrm{tr}}$ which is fixed by σ) is undecidable. *Algebraic function fields over algebraically closed fields.*

(presented by Eisenträger)

Theorem Let K be a finitely generated field over a field k ($\mathrm{char}(k) > 3$) of transcendence degree at least 1. If the transcendence degree of k over $\mathbf{F}_p$ is at least 1, then K is undecidable. If furthermore k is not algebraically closed then Hilbert's Tenth Problem for K is undecidable. *Elementary equivalence and isomorphism of finitely generated fields.*

(presented by Poonen)

Conjecture If K and L are finitely generated fields which are elementarily equivalent, then K and L are isomorphic.

Theorem Assume there exists a first-order formula $P(\vec{t}, u)$ such that for every discrete valuation ring $\mathcal{O}$ in any finitely generated field K (of fixed characteristic and fixed transcendence degree), there exists a tuple $\vec{t} \in K^n$ of parameters such that $\mathcal{O} = \{u \in K : P(\vec{t}, u)\}$. Then the conjecture is true.

Further developments

As a result of the results obtained during the discussion sessions several new collaborations have been set up or older collaborations have been rejuvenated, to continue the research initiated during the discussion sessions.

Several people have been interested in organizing a follow-up meeting to the meeting at AIM.