

COMPONENTS OF HILBERT SCHEMES

organized by

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Workshop Summary

Hilbert schemes are among the most fundamental moduli spaces in algebraic geometry. Despite their importance, many of their basic geometric properties remain a mystery, and there is a significant gap between well-understood Hilbert schemes and known pathologies. The mathematical aim of this workshop was to explore this gap. In particular, the workshop focused on the Hilbert scheme $\text{Hilb}^d(\mathbb{A}^n)$ of d -points in affine n -space, the Hilbert scheme $H_{d,g}$ of locally Cohen-Macaulay curves in projective 3-space, and small multigraded Hilbert schemes.

The participants included mathematicians who use a wide range of algebraic, geometric, computational, or combinatorial techniques to study Hilbert schemes. A surprisingly large number of the participants were not well-acquainted or had not met in person. A broader goal was thus to understand different perspectives, learn new methods, and prompt collaborations.

The first day of the workshop began with two survey talks. Robin Hartshorne provided an overview of the Hilbert scheme $H_{d,g}$ of locally Cohen-Macaulay curves in \mathbb{P}^3 and Daniel Erman reviewed the geometric properties of $\text{Hilb}^d(\mathbb{A}^n)$. Both speakers highlighted open problems and key examples. In the afternoon, we formed four learning groups. One group examined the explicit equations appearing in the Ekedahl-Skjelnes' blow-up construction of the smoothable component of $\text{Hilb}^d(\mathbb{A}^n)$. A second group analyzed the various possible definitions for the Hilbert-Chow morphism and, for the case of the Hilbert schemes of points, explained that these definitions coincide. A third group studied points in $\text{Hilb}^8(\mathbb{A}^4)$ which lie in the intersection of its two irreducible components. The fourth group constructed some examples and counterexamples of points and paths in $H_{4,-1}$. The activities of these subgroups were reported back to the entire group by Melanie Wood, Jason Starr, Dawei Chen, and Diane Maclagan respectively.

Tuesday started with a lecture by Mireille Martin-Deschamps on tools for exploring $H_{d,g}$. She emphasized the important numerical invariants, the Rao module, extremal curves, and liaison. Milena Hering then gave a talk introducing multigraded Hilbert schemes and relating them to other Hilbert schemes. The afternoon was devoted to five working groups. One attempted to compute the tangent spaces to the smoothable component at torus-fixed points on $\text{Hilb}^8(\mathbb{A}^4)$. Another group examined the moduli space of branch-varieties; introduced by Alexeev-Knutson, this variant of the Hilbert scheme replaces subschemes with finite morphisms. The next group looked for connections between Alexeev's moduli of stable toric pairs and certain multigraded Hilbert schemes. Yet another group contemplated the representation theory techniques used to understand the singularities of $\text{Hilb}^9(\mathbb{A}^8)$ by Kyungyong Lee. The final group investigated families of Rao modules. Verbal summaries of each group's insights were provided by Mark Huibregtse, Roy Skjelnes, Matthew Satriano, Dustin Cartwright, and Leila Khatami.

On Wednesday, Mike Stillman spoke about local equations for a neighbourhood of a torus-fixed point on a Hilbert scheme and demonstrated how these methods can reveal the structure of $\text{Hilb}^{3z+1}(\mathbb{P}^3)$ in *Macaulay2*. Following this, Ravi Vakil explained Murphy's Law for Hilbert schemes and presented several challenge problems. After lunch, we enjoyed an unexpectedly long problem session moderated by Izzet Coşkun and recorded by Li Li. The day finished with four working groups: one investigating a Grassmannian component with Hilbert function $(1, 4, 10, 6)$ of the $\text{Hilb}^{21}(\mathbb{A}^3)$, one constructing particular paths on the $H_{4,-1}$, one exploring examples arising from Mnev's universality theorem, and one probing the connectedness of the moduli stack of genus g curves. Laurent Evain, Scott Nollet, Mark Huibregtse, and Li Li relayed the progress of these short-lived groups.

Dawei Chen began on Thursday with a talk on the birational geometry of Hilbert schemes concentrating on the main component of $\text{Hilb}^{3z+1}(\mathbb{P}^3)$. Martin Olsson followed with a talk on Log geometry and how it can be used to distinguish certain components of moduli spaces. The afternoon was again devoted to five working groups. One determined generators for the effective, moving, and ample cones on $\text{Hilb}^2(\mathbb{P}^2)$. Another group created an assortment of examples of locally Cohen-Macaulay surfaces in \mathbb{P}^4 . The next group attempted to find the generic point on a Grassmannian component of $\text{Hilb}^{21}(\mathbb{A}^3)$ by understanding the deformations of a specific ideal. A separate group worked to understand the relationship between log geometry and the toric Hilbert scheme in a concrete example. The last group investigated some determinantal equations for Hilbert schemes of points. Izzet Coşkun, Enrico Schlesinger, Anthony Iarrobino, Jesse Kass, and Erik Carlsson verbally summarized the developments for each group.

The final day started with Jason Starr speaking about pseudo Hilbert schemes and their potential applications. Mathias Lederer next described the combinatorial objects labelling the irreducible components of certain Gröbner strata of $\text{Hilb}^d(\mathbb{A}^n)$. Once more, the afternoon was dedicated to five working groups. The first sought to write down the equations for four lines in \mathbb{P}^3 degenerating to a multiple line. The second recounted the connections between moduli of quiver representations, Hilbert schemes, and nilpotent matrices. The third endeavoured to construct toric degenerations of some Hilbert schemes. The fourth investigated specific examples of Gröbner strata in an effort to generalize results from Lederer's talk. Finally, the fifth group strived to exhibit low-dimensional irreducible components of $\text{Hilb}^d(X)$ where X is a singular curve. The workshop concluded with short group reports from Robin Hartshorne, Roy Skjelnes, Daniel Erman, Margherita Roggero, and Jesse Kass.

By general consensus, the afternoon working groups were particularly worthwhile. Several participants said that this was the most enjoyable conference they had attended recently. We are grateful to AIM for providing such pleasant working conditions, and to our participants for sharing so generously of their knowledge and ideas.