Homological mirror symmetry and multigraded commutative algebra

organized by
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Workshop Summary

Introduction

Breakthrough work of Hanlon-Hicks-Lazarev in 2023 gave proofs of several conjectures in multigraded commutative algebra and toric geometry using techniques in symplectic geometry and homological mirror symmetry. This work, and a flurry of subsequent work inspired by it, led to an AIM Hot Topics Workshop in 2023 whose goal was to continue to develop this burgeoning interface between commutative algebra, algebraic geometry, and symplectic geometry. That workshop was a major success; for instance, a highlight of the collaborations that grew out of it is a recent paper of Ballard-Berkesch-Brown-Cranton Heller-Erman-Favero-Ganatra-Hanlon-Huang that gives a subtle reinterpretation and positive resolution of a major conjecture in algebraic geometry and derived categories. The key innovation in this paper was a triangulated category called the *Cox category*, which is "glued" from derived categories of birational models of a given toric variety.

This AIM workshop (Homological Mirror Symmetry and Multigraded Commutative Algebra) was a follow up on the 2023 AIM Hot Topics Workshop and the work that grew out of it. More specifically, the theme of the workshop was further investigations into the newly discovered Cox category and the surrounding circle of ideas. An especially exciting aspect of this workshop was the special focus on developing and implementing algorithms in the symbolic algebra system Macaulay2, which will lead to a great increase in the capacity to explore Cox categories and related constructions with computer experimentation. We expect that the software developed during this workshop will be useful to researchers in commutative algebra, toric geometry, symplectic geometry, and beyond.

After generating ideas as a group, the workshop participants eventually settled on the following six projects:

(1) Implementing homological mirror symmetry and the Fukaya category in Macaulay2
(2) Producing Kawamata's exceptional collections for toric varieties in Macaulay2
(3) Implementing toric stacks in Macaulay2
(4) Investigating properties of Hanlon-Hicks-Lazarev resolutions
(5) Exploring mutations of Lagrangian skeleta and their relation to algebro-categorical mutation
(6) Developing a geometric interpretation of the Serre Functor on the Cox category

We describe these projects, and the future plans inspired by them, in detail below.

Homological mirror symmetry and Fukaya categories in Macaulay2

The main goal for this project was to add the wrapped Fukaya category into Macaulay2 in order to compute small examples for mirrors to toric varieties and perform small modifications of the stop data of the FLTZ skeleton. The first step was computing Hom for Bondal-Thomsen elements. The group introduced a data structure for the wrapped Fukaya category on a torus by storing hyperplanes and the hairs on the faces of the cell complex of the hyperplane arrangement for certain classes of examples. An issue was that the rules for wrapping the corresponding Lagrangian are not easy to describe in terms of this combinatorial data of hyperplanes and hairs. The group is optimistic that that they will be able to resolve this with further effort, and they expect to continue work on this project at a future Macaulay2 workshop.

Kawamata's Exceptional Collections for Toric Varieties in Macaulay2

This project started with an intent to write a Macaulay2 implementation of Kawamata's algorithm, as it appears in the work of Ballard-Favero-Katzarkov. During the project, the group realized that they could obtain exceptional collections through a mutation process in the Cox category. As an example, they computed full exceptional collections for the projective stack $\mathbb{P}(1,1,3)$ from a known exceptional collection of the Hirzebruch surface $\mathbb{F}_3$. They implemented methods for performing Cox category mutations in Macaulay2 and tested them for $\mathbb{F}_3$. In the future, they aim to generalize this example to a Kawamata-like algorithm in any Cox category.

Toric stacks in M2

This project aims to build functionality for computing with toric stacks in Macaulay2. The main goals/emphasis are two-fold: (1) build out basic functions (as has been done for toric varieties, for example) that enable computing examples to gain intuition and familiarity with toric stacks, and (2) implement some more specialized functionality, including recovering toric stacks and morphisms from cones in the GKZ fan. The group expects to continue development of the package over the next year.

Properties of HHL resolutions

The goal of this project was to investigate properties of families of HHL resolutions. Our group has been particularly interested in classifying which examples of toric varieties have an identity point that is projectively normal. The question can be phrased concretely as: when does the minimal free resolution of the normalization of the ring of a point have just one copy of the Cox ring in homological degree 0? The group first computed several small examples by hand, gaining comfort with the computations for Hirzebruch surfaces and weighted projective lines. Utilizing work of Hanlon-Hicks-Lazarev and Favero-Sapronov, the problem reduces to computing the strata of the torus and their compactly supported homologies. The group plans to continue this work, with the next family of examples being higher dimension weighted projective spaces via Jay Yang's Macaulay2 code that computes the HHL resolutions.

Mutations and Skeleta

The mirror to a toric variety has a combinatorially described (relative) skeleton called the FLTZ skeleton. From the perspective of symplectic geometry, skeleta are more naturally obtained from a geometric structure called a 'symplectic Lefschetz fibration' via a process of 'handle attachment.' We call a skeleton arising from a Lefschetz fibration via this construction a 'Lefschetz skeleton.' Given a Lefschetz skeleton, it's a theorem that the partially wrapped Fukaya category admits a full strong exceptional collection (FSEC) and mutation has a geometric description in terms of Dehn twists of vanishing cycles. The question is: when is the FLTZ skeleton (or the Cox skeleton) obtainable from a Lefschetz skeleton via a sequence of 'moves?'

The group's suggested approach to this problem is to construct a larger skeleton, called the 'topological skeleton,' which is claimed to be a Lefschetz skeleton. Hopefully, parts of this topological skeleton can be removed to obtain the Cox skeleton (or the FLTZ skeleton for a toric mirror) in a way that preserves the property of being Lefschetz. Of course, one cannot hope to do this in general since some toric varieties do not admit FSECs. In some sense, this is the 'reverse' of the Kawamata construction, in the sense that it starts with a 'large' exceptional collection and seeks to find an exceptional collection for a toric variety by 'localizing away' extraneous elements of the collection. The next step is to understand geometrically a condition under which one can remove parts of the topological skeleton whilst preserving the Lefschetz condition.

The Serre functor on the Cox category

Given a toric variety X, aforementioned work of a host of authors (Ballard-Berkesch-Brown-Cranton Heller-Erman-Favero-Ganatra-Hanlon-Huang) describes a triangulated category—called the Cox category of X—that encodes the homological algebra of all of the toric varieties in the secondary fan of X. Their main result is that this category has a full strong exceptional collection. As a formal consequence, one concludes that the Cox category of X is equipped with a Serre functor, which is a categorifical generalization of the classical notion of Serre duality in algebraic geometry. The goal of this group was to explicitly describe this Serre functor and explore some applications. The group was able to prove that the Serre functor is given, roughly speaking, by convolution with the dual of an explicit resolution of the diagonal for X. The group expects to continue exploring applications of this calculation, e.g., a version of Bondal-Orlov reconstruction for the Cox category.