Stability, hyperbolicity, and zero localization of functions

organized by
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Workshop Summary

1 Overview

This workshop, sponsored by AIM and NSF, was devoted to the emerging theory of stability and hyperbolicity of functions. These notions are well known in the univariate setting, where stability means that all zeros lie in the left-half plane and hyperbolicity means that all zeros are real [CM, Post, Fisk, Rah]. The multivariate generalizations go back to 1950s and are being actively explored now [BB2,WagB]. Among recent applications of multivariate stability and hyperbolicity are the proof of Johnson's conjectures on mixed determinants, new proofs of van der Waerden and Schrijver-Valiant type conjectures, and the resolution of several conjectures on negative dependence in discrete probability theory [Gur2, Gur3, Wag2, BBS, BB1, BB2, BBL]. For surveys of these and further developments, see http://www.math.hawaii.edu/∼tom/mathfiles/czdssurvey.pdf and
http://www.ams.org/journals/bull/2011-48-01/S0273-0979-2010-01321-5/home.html

The workshop participants included experts in real and complex analysis, orthogonal polynomials and special functions, combinatorics and probability, statistical mechanics, as well as matrix and operator theory. While some of the participants knew each other and some even collaborated previously, many also met for the first time during the workshop. Thus, a broader goal of the workshop was to promote interaction and future collaboration among researchers in distinct but interrelated fields.

The following main open problems were initially selected as focal points of this workshop:

(1) The Bessis-Moussa-Villani (BMV) conjecture [BMV] originally arose in the theory of quantum mechanical system. It states that $tr(\exp(A-tB))$ is the Laplace transform of a positive measure for any two positive definite matrices A and B. It can be restated as a conjecture about the positivity of all coefficients of specific polynomials.
(2) The Mason conjecture [Mason] states the (ultra-)log concavity of the f-vector of a matroid or, equivalently, of the sequence of coefficient of the so-called independent set polynomial of that matroid.
(3) Lee-Yang problems of statistical mechanics and multivariate stability/ hyperbolicity [LY1]: Phase transitions in statistical mechanics can be analyzed via the zeros of the corresponding partition functions. However, there is no general theoretical framework describing zero distribution of partition functions in statistical mechanics (Ising, Potts, and other models).
(4) Pólya-Schur problems [BB1,BB2]: classification of linear (or non-linear) preservers of polynomials and entire functions in one or several variables with prescribed zero sets.

In two exciting recent developments, Conjecture 1 (BMV) and the weaker form of Conjecture 2 (Mason's log-concavity) were proved shortly before the workshop, the BMV conjecture by Herbert Stahl [Stahl] and the Mason conjecture by Matthias Lenz [Lenz], using the work of Huh and Katz [HuhKatz]. As a result, the goals of the workshop per these conjectures shifted accordingly: to understand their recent proofs and to explore how these new mathematical ideas can be explored further.

2 Timeline of the workshop

The workshop began with two introductory talks by Olga Holtz and Petter Brändén. Holtz' talk surveyed problems of zero localization in analytic number theory, concentrating on the Riemann $\zeta-$ and Ξ- functions, the theory of orthogonal polynomials, and various positivity questions, including the BMV and Mason conjectures. Brändén's talk was devoted to Pólya-Schur problems, focusing on operators that preserve polynomials with zeros in a given domain and extending to applications to combinatorics and statistical mechanics.

During the Monday afternoon problem session, the list of open problem collected prior to the workshop was discussed and augmented by further problems suggested by the participants. Several problems were selected for work in groups on Tuesday afternoon.

Alan Sokal and Robin Pemantle each gave a series of two talks, on Tuesday and Wednesday morning. Sokal's series was focused on special generating functions that arise in statistical mechanics as well as in classical analysis and several other fields. Those include the so-called partial theta function $\sum_{n=0}^{\infty} x^n y^{n(n-1)/2}$ and the deformed exponential function $\sum_{n=0}^{\infty} x^n y^{n(n-1)/2}/n!$. Sokal formulated several conjectures about the zero distribution of such functions and showed how to prove them in the case of the partial theta function.

Robin Pemantle spoke about two main subjects: asymptotics of multivariate generating functions and negative dependence. Pemantle discussed a number of combinatorial problems in which rational generating functions arise, whose denominators have factors with certain singularities. He showed how to compute the asymptotics of the coefficients of such a generating function in generic and non-generic situations. These computations require some topological deformations as well as Fourier-Laplace transforms of generalized functions. These results apply to specific combinatorial problems, such as Aztec diamond tilings and cube groves.

Pemantle's second talk was centered around negative correlation in probability theory [BBL,Pemantle,Wag2], including strong negative correlation property called the Rayleigh condition and the strong Rayleigh condition. He described how these notions naturally leads to a more general, matroidal, setup, how they are related to zeros of multivariate polynomials, and how Mason's Conjecture arises in this context, ending his talk with further open problems.

Thursday morning, Pierre Moussa gave a talk on the history and various aspects of the BMV conjecture. Moussa began his talk with a discussion of the original measure-theoretic version of the BMV conjecture [BMV] and then continued to discuss an equivalent matrix formulation of the conjecture [LS,Mou]: *Let $n \in N$ and let A, B be positive definite Hermitian $n \times n$ matrices. Then, for any $k \in N$, the polynomial $t \mapsto tr(A + tB)^k$ has only nonnegative coefficients.*

The last part of Moussa's talk was devoted to the recent measure-theoretic proof of the BMV conjecture by Herbert Stahl [Stahl]. The audience indicated strong interest in understanding that proof better. Various proposals were made, such as going through the proof for relatively simple subclasses of the pairs (A, B).

Shmuel Friedland subsequently spoke about recent results and open problems on graph matchings. His talk was divided into three parts: formulas for k-matchings in bipartite graphs, including the corresponding generating functions (some of which are known to be hyperbolic), upper bounds on permanents and pfaffians, and lower bounds on permanent and pfaffians. Friedland included a list of open problem as well.

Friday morning, Petter Brändén reported on very recent progress on one of the Pólya-Schur problems: characterizing operators that preserve polynomials with zeros in a given strip.

Tuesday through Friday afternoons were devoted to work in groups on the open problems formulated on Monday as well as a few further problems added at a later stage. We report progress in these direction in our next section.

3 Progress made during the workshop

A list of the open problems that were considered at the workshop may be found at the American Institute of Mathematics Problem Lists: http://aimpl.org/hyperbolicpoly/ . Most of the problems or conjectures listed there were submitted by the participants prior to the start of the workshop, while others were formulated during the workshop.

The afternoon group discussions and exchange of ideas have proved to be particularly fruitful. Indeed, they included not only clarifications of the questions under consideration, but also provided a forum for assessing different strategies and approaches. We mention several instances where that group work has led to (full or partial) solutions to some of the problems.

Conjecture 8.1 (See http://aimpl.org/hyperbolicpoly/8/). Let $p_n(z) = \sum_{k=0}^{n}(-1)^k(2k+1)z^{k(k+1)}$, where $n = 2, 3, 4, \ldots$. A. Vishnyakova conjectured that p_n has no real zeros for even n, and exactly two real zeros for odd n. The group working on this problem included Vishnyakova, Pemantle, Katkova, Edwards, Craven and Csordas.

This conjecture was shown to be true when n is even. The proof was based, in part, on the following beautiful formula

$$\sum_{m=0}^{\infty}(-1)^m(2m+1)z^{m(m+1)/2} = \prod_{m=1}^{\infty}(1-z^{2m})^3 \quad (|z| < 1).$$

(See, for example, the Appendix, p. 164, A I #54.1 in [PolyaSzegoII].) When n is odd, the conjecture appears to be still open.

Problem 7.2 (See http://aimpl.org/hyperbolicpoly/7/). J. Haglund raised the following question. If $\alpha > 0$, does the Fourier transform

$$F_\alpha(z) = \int_0^\infty \Phi^\alpha(t)\cos(zt)dt$$

have any non-real zeros? Here the kernel, Φ, denotes the Jacobi theta function whose Fourier transform is the Riemann Ξ–function (cf. http://aimpl.org/hyperbolicpoly/7/). High-precision calculations, using the moments of Φ, show that when $\alpha = 20$, some of the Jensen polynomials associated with F_{20} have non-real zeros and whence F_{20} possesses non-real zeros. We are still trying to understand this curious phenomenon. The participants who worked on this problem included Ki, Haglund, Tyaglov and Csordas.

Problem 6.3 (See http://aimpl.org/hyperbolicpoly/6/). One of the fundamental questions in the theory of orthogonal polynomials and their generalizations is whether the polynomials in question satisfy some recurrence relations. Pierre Moussa asked if orthogonal polynomials with constraints can be related to each other via a recurrence relation. He mentioned some special cases where this problem has been solved. Shmuel Friedland proposed to consider the operator P^*TP, where T is the operator of multiplication by the independent variable and P is the projector on the subspace S. Then the usual Lanczos' algorithm applied to P^*TP gives us certain recurrence relations for orthogonal functions in S. The work on this problem by Derevyagin, Driver, Moussa, Friedland and Tyaglov is still in progress.

Problem 4.3 (See http://aimpl.org/hyperbolicpoly/4/). Chudnovsky and Seymour proved [CS] that the generating polynomial of independent sets in a claw free graph has only real zeros. This generalizes the univariate consequence of the multivariate Heilmann-Lieb [HL] theorem for matchings in graphs. Brändén, Leander, Visontai and Sokal worked on trying to generalize the Chudnovsky-Seymour theorem to a multivariate setting. Promising ideas and approaches emerged and it is currently work in progress.

Problem 8.3 (See http://aimpl.org/hyperbolicpoly/8/). The deformed exponential function is approximated by the polynomials $P_n(x, w) = \sum_{k=0}^{n} \binom{n}{k} x^k w^{k(n-k)}$. Sokal and Holtz proposed two different arguments that prove the conjectured separation of zeros for w real and derived a new recurrence relation connecting P_n and P_{n-1}, which may be useful to address the general case.

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