Analysis on the hypercube with applications to quantum computing

organized by

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Workshop Summary

Introduction

The workshop was devoted to several challenging open problems, such as Aaronson–Ambainis conjecture, on the hamming cube raised in theoretical computer science which have some implications in understanding of the advantage of quantum algorithms over the classical ones. The goal of the workshop was to bring young mathematicians from different fields including harmonic analysis, probability, computer science, to introduce all the necessary notions required for understanding these challenging problems, to describe their connection to Fourier analysis, and to discuss known techniques and methods used in the proofs of partial results to these problems. Each day of two talks were presented on the topic related to the Hamming cube. Participants were divided into 5 groups, and each group worked on a different problem. Here is the short report from each group.

Aaronson–Ambainis conjecture.
We considered the problem for two afternoons.

We know that possible counterexamples to the conjecture, whose existence we would like to refute, are tribes-like. It is moreover known that symmetric functions obey the AA statement. We generalized this result to block-symmetric functions (and slightly more), that is

$$f = F(G(x_1, \ldots, x_m), \ldots, G(x_{n-m+1}, \ldots, x_n))$$

with F, G symmetric. This class of functions include the tribes function, but not tribes-like functions. Ideas only slightly extend known techniques. Next interesting aim is to consider $f = F(G_1(x_1, \ldots, x_m), \ldots, G_k(x_{km-m+1}, \ldots, x_{km}))$ with F, G_i symmetric. This family is much closer to include tribes-like functions.

While decision trees (DT) are useful when studying AA conjecture, we determined that random DTs are improbable to deliver additional utility, as AA mostly concerns with average-case efficiency of the DT.

On the perhaps negative side, we examined the following conceivable instantiation of AA conjecture $\|f - \mathbb{E}[f]\|_2^2 \leq \deg(f)^{O(1)} \max_i \|D_i f\|_2 \|f\|_\infty$. We found out that homogenuous polynomials with random ± 1 coefficients are (at least) saturating this inequality, hinting that perhaps a clever choice of functions could serve as better (counter)examples. Next natural aim is to invent somewhat-symmetric functions with small $\|f\|_\infty$.

Markov–Bernstein type inequalities.
At our group we tried to justify the inequality

$$\|\Delta f\|_p \leq C_p d \, \|f\|_p \tag{1}$$

for $1 < p < \infty$ and functions f of degree at most d, i.e. of the form

$$f(x) = \sum_{S\subseteq[n],|S|\le d} \hat{f}(S)x^S \qquad x \in \{-1,1\}^d.$$

The inequality (1) is conjectured to hold for all real or even Banach valued functions f. During the workshop we made some partial progress by justifying (1) for those f which take values in $\{-1,0,1\}$. This is a non-trivial results since the L^∞ endpoint of (1) is not true even for Boolean-valued functions.

Bohnenblust–Hille inequality.
We worked on the Bohnenblust–Hille inequalities on hypercubes. The inequality states that for any $d \ge 1$, there exists $C_d > 0$ that depends only on d, such that for all $n \ge 1$ and all homogeneous polynomials $f = \sum_{|S|=d} \widehat{f}(S)x_S$ of degree d on $\{-1,1\}^n$, we have

$$\left(\sum_{|S|=d} |\widehat{f}(S)|^{\frac{2d}{d+1}}\right)^{\frac{d+1}{2d}} \le C_d\|f\|_\infty.$$

If we denote by BH(d) the best constant C_d, then it is known that (Defant–Mastyo–Prez: On the Fourier spectrum of functions on Boolean cubes, Math. Ann. 2019) there exists a universal constant $C > 0$ such that $\mathrm{BH}(d) \le C^{\sqrt{d\log d}}$ for all d. This inequality has a long history and was not studied on hypercubes first. This variant on hypercubes has found many applications, e.g. in the learning theory.

During the AIM workshop, we tried to improve the Bohnenblust–Hille constant BH(d). The aim is to show that BH(d) is at most of the polynomial growth or is even bounded by some universal constant. The best known result is $\mathrm{BH}(d) \le C^{\sqrt{d\log d}}$, and their idea is to first prove an inductive inequality $\mathrm{BH}(d) \le C(d,k)\mathrm{BH}(k)$ for some $C(d,k)$ and for all $1 \le k \le d$, and then applies it to special k. We tried to choose different k's, but this did not lead to anything better. By checking carefully their proof of this inductive inequality, we focused on two places to improve the constants $C(d,k)$, then hopefully BH(d).

The first place is the so-called Blei's inequality, as a generalization of the well-known Littlewood's 4/3 inequality. The proof is a clever combination of Hlder's inequality and Minkowski's inequality, and the constant is clearly sharp. However, maybe one can find a better way in using this inequality that allows for improving the constant $C(d,k)$, and this is one of the main approaches we tried in the workshop. First, a standard argument is to symmetrize the Fourier coefficients $\widehat{f}(S)$ before applying Blei's inequality, but we noticed that the estimates can be slightly better if we do not employ the symmetrization. This did not improve the upper bound of BH(d). The second observation is that when applying Hlder's inequality, a different choice of the exponent may yield a great improvement in the estimates. The idea is that one can use Hausdorff–Young inequality to unify the argument of Plancherel identity plus the moment comparison inequalities, with a better constant of 1. Unfortunately, we only imagined this variant of Blei's inequality and have not designed it successfully. This will be further explored.

The second place is polarization. This is a classical tool to pass from multilinear forms to polynomials. Here the polarization inequality is on the cubes $[-1,1]^n$ and the constant is known to be sharp. This constant is very badly behaved and is one of the main obstacles to improving BH(d). We do not have too many thoughts on polarization. Maybe we need to use some special structures of hypercubes $\{-1,1\}^n$ and find replacements of polarization here. This is to be investigated.

Apart from improving the Bohnenblust–Hille constant $\mathrm{BH}(d)$, we may also consider the vector-valued generalization of Bohnenblust–Hille inequalities on hypercubes in the future.

On the sharp constant of Khintchine type inequality for Boolean functions .

Our group studied the problem of determining the sharp constant $C_{p,q} > 0$ such that

$$\forall f : \{-1,+1\}^n \to \mathbb{R}, \quad f_q \leq C_{p,q}^{d(f)} f_p, \quad \text{where } p < q \leq 2,$$

and $d(f)$ is the degree of f. Instead of working on the general case, we turn to focus on the homogeneous degree setting. In particular, for $q = 2, p = 1$, we consider degree 2 homogeneous function $f(\mathbf{x}) = \sum_{i<j} a_{ij} x_i x_j$, where $a_{ij} \in \mathbb{R}$. The conjectured sharp constant $C_{1,2} = \sqrt{2}$. The degree 1 case is just standard Khintchine's inequality and the sharp constant was proved to be $\sqrt{2}$ long time ago by Szarek. We explored some previous results using complex hypercontractivity technique by Ivanisvili and Tkocz, where in the homogeneous degree k case, the best constant they obtained is $\sqrt{e}$. It seems hard to improve this to $\sqrt{2}$ using the similar approach. We also revisited the proof details of $d(f) = 1$ case by Szarek in 1976, where the idea is to reformulate the problem as constraint optimization question, then obtain a characterization of the minimizer by some careful analysis. It seems promising that this idea can be adapted to general homogeneous odd $d(f)$ case. In the future, we plan to understand all the details of this proof and carry out the analysis to higher degree case.

The sensitivity theorem.

We worked on sensitivity theorem which says that the sensitivity of any Boolean function of degree d is at least square root of d. This theorem was recently solved by Huang Hang and it completes the picture of all complexity measures introduced for Boolean functions in computer science, namely all these different complexity measures are polynomials related to each other. The goal of our group was to find out if there is an analog of sensitivity theorem for real valued functions on the hypercube of degree d. We looked at different notions of sensitivity and we looked for an inequality which, when applied to boolean functions, would coincide with sensitivity theorem. We tried to use LittlewoodPaley type inequalities and some other harmonic analytic techniques to verify several our conjectured inequalities.