Integral Closure, Multiplier Ideals and Cores

organized by
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Workshop Summary

The goal of our workshop was to bring together active senior researchers and promising recent PhD students working at the interface of Commutative Algebra and Algebraic Geometry to interact on a focused set of problems about 'integrality.' To be specific, we planned to study questions grouped around the following cluster of topics, which are related in unexpected and surprising ways and offer remarkable computational challenges:

- integral closure and complexity
- multiplicities
- cores of ideals and modules
- adjoint and multiplier ideals
- Briançon-Skoda type theorems.

An extensive e-mail correspondence resulted in a list of problems of interest to the selected participants; this list helped us coordinate the speakers and plan the activities of the upcoming workshop. Still, we kept an open mind to allow for a great deal of flexibility throughout the meeting. Thirteen of the thirty-three participants were young researchers within five years from receiving their doctorate: six of them are currently postdocs (Caviglia, Crispin, Epstein, Fouli, Hong, Yuen) and two are finishing graduate students (Lee and Validashti). We also paid attention to bringing scholars from liberal arts institutions without doctoral programs (Reed College, SUNY at Potsdam and Southeastern Louisiana University) and encouraged the participation of female researchers (seven attended out of eight invitees).

The workshop was structured following the tradition crystalized at AIM. Each morning we had two introductory talks given by David Eisenbud, Mel Hochster, Craig Huneke, Steve Kleiman, Kyungyong Lee, Joe Lipman, Shunsuke Takagi, Ngo Viet Trung, Bernd Ulrich, Kei-ichi Watanabe. Both Eisenbud and Huneke's lectures initiated lively group discussions in the afternoons of each talk. A group led by Eisenbud grew out of the effort to understand the asymptotic regularity of powers of ideals. More precisely it focused on a recent paper by Römer relating an asymptotic linear bound on the regularity of powers to the bigraded generic initial ideal of the ideal defining the Rees algebra. The outcome of the group discussion was a thorough understanding of the paper and considerable simplifications of several arguments. The talk by Huneke instead discussed a series of open problems related to the integral closure of ideals: these ranged from the Eisenbud-Mazur conjecture, to the growth of symbolic powers in mixed characteristic, and packing properties in linear programming via blowup algebras. These conjectures were analyzed in depth in a full length afternoon discussion which was very well attended.

Apart from these short-lived groups, additional ones were formed after a lively initial discussion, moderated by Irena Swanson, which set the tone for the rest of the workshop.

We were impressed by the smoothness with which the groups were formed. In what follows we give an account of the issues addressed by each group. Detailed group reports on the activities of the previous day opened each afternoon working session.

Equisingularity and j-multiplicity.
The concept of integral dependence of ideals and modules is essential in intersection theory and the theory of multiplicities. Rees' theorem from the sixties provides the main connection: two zero-dimensional ideals $J \subset I$ have the same integral closure if and only if they have the same Hilbert-Samuel multiplicity. Many generalizations and extensions of this result have appeared in the literature. On the algebraic side we recall the work of Achilles, Flenner and Manaresi, who introduced the notion of j-multiplicity as a generalization of the Hilbert-Samuel multiplicity to ideals which are not necessarily zero-dimensional. On the geometric side, and more precisely in equisingularity theory, Gaffney, Kleiman and Teissier have used the Buchsbaum-Rim multiplicity (and generalizations thereof) to show integral dependence of modules that are often images of Jacobian matrices. During the workshop, there were two general talks related to these topics: one by Kleiman (from the equisingularity side) and one by Ulrich (from the algebraic side).

The group discussion started from establishing some sort of basic common program and agreeing on desirable properties of a new unified 'multiplicity' for pairs of submodules $M \subset N$ of a free module, that do not necessarily have finite colength. For example, this multiplicity should coincide with the Buchsbaum-Rim multiplicity when the modules have finite colength in the free module; two modules should have the same integral closure if and only if their relative multiplicity vanishes; furthermore, the multiplicity should satisfy an additivity formula for triples of submodules, and the principle of specialization. The group also set out to compare the algebraic approach based on local cohomology and length with the geometric one that uses Segre classes. This is a first important step towards advancing the theory and finding unified, more general results that can be applied to study equisingularity of not necessarily isolated singularities. Several promising calculations were carried out for this purpose using algebraic methods as well as geometric ones. These efforts are ongoing.

Integral closure via colon operations.
Another main problem that needs to be addressed and eventually solved is of a computational nature; that is, we need to find an 'effective' method to construct the integral closure of an ideal. The only known strategy uses Rees algebras, but this is a very expensive procedure since the integral closure of all powers of I are computed simultaneously. In 1968, Burch had shown that socle elements provide, in general, new elements integral over a given ideal. Her result applies to ideals of finite projective dimension in a local ring which is not regular. Considerably later, Corso, Polini and Vasconcelos extended this idea and showed that ideals of the form $Q\colon \mathfrak{m}$, where Q is a complete intersection ideal and $\mathfrak{m}$ is the maximal ideal of R, are integral over the ideal Q in a broad context which includes regular rings as well. There has been extensive work (along these lines but with a variety of different techniques) by Corso and Polini, Polini and Ulrich, Goto and Sakurai, Wang. There was no formal talk about this approach to constructing integral closures since there was already a solid group of people familiar with the subject and ready to dive directly into the issues. This group started by comparing the various homological and ideal theoretic methods used thus far in this problem. It was agreed that it is essential to understand well the behavior of generalized socle elements in regular local rings. For this purpose Huneke proposed to study the number

$\sup_Q \max\{s \mid Q\colon \mathfrak{m}^s \subset \overline{Q}\}/\operatorname{ord} Q$. Calculations done by the participants suggested that this number is 1 if the ring is regular of dimension 2, whereas it is ∞ if the ring is regular of dimension at least 3. Another direction that has sparkled great interest is the variety of behavior of these quotients in rings of dimension 1. These investigations are still in progress. Some of the members of the group will meet again in a forthcoming workshop to be held at Banff Research Station in the Summer of 2007. Another meeting in Japan is anticipated in March 2008.

Adjoint and multiplier ideals.
The Briançon-Skoda theorem is a purely algebraic statement which was first proved using analytic methods; it says, in one of its simplest forms, that if R is a d-dimensional regular local ring then the integral closure of I^d is contained in any reduction of I. Lipman was the first to introduce and study adjoint ideals for regular local rings from an algebraic point of view. His goal was to better understand the Briançon-Skoda theorem and to extend Zariski's theory of integrally closed ideals in two-dimensional regular local rings. He proved that the adjoint ideal of I^d is contained in every reduction of I, thus improving the Briançon-Skoda theorem. On the other hand, multiplier ideals are integrally closed ideals that have been defined in complex algebraic geometry using log resolutions. When adjoint and multiplier ideals are both defined the two notions agree. Lipman and Watanabe proved the remarkable fact that every integrally closed ideal of a two-dimensional regular local ring can be realized as a multiplier ideal. Briançon-Skoda type theorems can also be obtained using the theory of tight closure of Hochster and Huneke. An important concept in tight closure theory is that of a test ideal; the analogy between this characteristic p notion and multiplier ideals has been developed with great success by Hara, Takagi, Watanabe and their coauthors.

During the workshop, we had introductory talks by Lipman, Lee, Watanabe, and Takagi. One of the questions that was raised by Huneke dealt with the syzygies of integrally closed ideal. This question was motivated by the talk by Lee on his recent ground-breaking work with Lazarsfeld, in which they show that minimal syzygies of multiplier ideals are rather special (have a low order of vanishing). Although their result suffices to conclude that there exist integrally closed ideals in dimension greater than two which are not multiplier ideals, it is still open how much of the special properties of multiplier ideals is shared by the whole class of integrally closed ideals. Takagi and Watanabe spoke instead about F-jumping numbers and F-thresholds in rings of prime characteristic. These numbers are characteristic p analogues of the corresponding notions in birational geometry. They can be viewed as a measure of singularity and provide a powerful tool for studying local rings. In particular, if R is a Noetherian local ring of prime characteristic and $\mathfrak{a} \subset \sqrt{I}$, then Watanabe introduced the number $c^I(\mathfrak{a}) = \lim\limits_{q\to\infty} \dfrac{\max\{r \mid \mathfrak{a}^r \not\subset I^{[q]}\}}{q}$, where q is a power of the characteristic of the ring. He proposed several remarkable conjectures and related them to each other. These open problems have been the basis of several afternoon discussions. Watanabe also provided a new characterization of integral closure, reminiscent of the definition of tight closure: if R is Cohen-Macaulay of dimension d and $J \subset I$ are two ideals, he showed that $c^J(I) = d$ if and only if $I \subset \overline{J}$. This result stirred quite a bit of attention and the afternoon was spent in a successful effort, led by Huneke and others, to prove Watanabe's result without the Cohen-Macaulay assumption on the ring. Watanabe also discussed how his techniques provide an application of the result of Wang that he just learned during the workshop; he was still cautious about his 'fresh' proof, but it passed the scrutiny of the group.

Throughout the week, members of this group focused on computational aspects as well. Computing multiplier or adjoint ideals, even for specific classes of ideals, is a difficult task. Progress was made in doing this for defining ideals of affine monomial space curves. One of the problems in computing adjoint ideals stems from the fact that it is not clear a priori which divisorial valuations are needed. This question was addressed in particular by Teitler and Swanson. Another unanswered question that received some attention is whether one can use alterations, as opposed to desingularization, to calculate multiplier ideals. Various smaller groups were formed around these issues and further developments are expected in the near future.

Finally, we would like to express our appreciation for the support from AIM and the NSF. In particular, a heartfelt thanks goes to the entire staff at AIM for tirelessly guiding and assisting us throughout the various stages of our planning. All participants benefitted tremendously from such a dynamic and singular type of workshop.