Arithmetic intersection theory on Shimura varieties

organized by
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Workshop Summary

Introduction

The purpose of this workshop was to bring together experts from around the globe to study and expand on problems arising in the Kudla program, which seeks to establish an explicit connection between the enumerative (arithmetic) geometry of Shimura varieties and special values of Eisenstein series, and other automorphic forms, with applications to many problems including the arithmetic Gan-Gross-Prasad conjectures, as well as the Bloch-Beilinson conjecture.

In recent years, there has been tremendous progress on various parts of the program, including a proof of the non-singular cases of the Kudla-Rapoport conjecture, a proof of various forms of the Arithmetic Fundamental Lemma and Arithmetic Transfer, a systematic construction of Green's currents for special cycles on Shimura varieties, developments in equicharacteristic formulations of the main conjectures, and the introduction of methods from derived algebraic geometry.

The intent of the organizers was to bring all these new developments to a wider (and younger) audience, and to foster open and generous sharing of ideas for current outstanding problems in the area, as well as to engage in some more speculative ventures with an eye towards future developments.

With AIM's wonderful support, we think these goals were more than met, and we believe that there will be a substantial scientific contribution to the field as a consequence of the conversations and discourse arising directly (and indirectly) from the workshop.

Morning talks

The morning talks were of two different flavors: At the beginning of the weeek, we had introductory ones by more senior attendees: topics covered included derived algebraic geometry, relative trace formula methods in the proofs of AFL type results, the use of Quillen's superconnections in constructing Green's currents, the theory of shtukas and their moduli spaces over function fields. Towards the latter half, we had talks by some of the younger participants, primarily graduate students, focusing on their dissertation work, and giving them an opportunity to share their work with the other attendees.

Problem groups

We started on Monday with 44 posed problems. The organizers selected eleven (11) of these for voting on Tuesday, and by the end of the week, five of them ended up with a substantial base of interest among the attendees. All of these problems have a very novel flavor, and progress in any of them will be of great scientific interest.

Exceptional groups and (higher) Siegel–Weil Formulas.

Motivated by the recent development of the theory of modular forms on exceptional groups, as well as the theory of exceptional theta correspondences, this working group discussed questions on how to formulate arithmetic (resp. higher) Siegelâ€“Weil formulas on Shimura varieties (resp. moduli of Shtukas) for exceptional groups. We focused on cases of dual pairs of the form $\mathrm{G}_2 \times \mathrm{H}$ where H is either SO_3, PU_3, PGSp_6, or F_4, which can be studied uniformly as "automorphisms of cubic norm structures." One motivation for this is that there are Siegel-Weil type formulas known in these settings, going back to work of Gan on E_8 on the pair $\mathrm{G}_2 \times \mathrm{F}_4$.

To formulate the corresponding moduli problems, note that Fourier coefficients of modular forms on G_2 are naturally parameterized by integral binary cubic polynomials $p(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$, or stated differently, by cubic rings $R = \mathbb{Z}[t]/(p(t,1))$ over $\mathbb{Z}$. With this in mind, we formulated moduli problems for special cycles in both the number field and function field settings. Motivated by formulas of Gan–Gross–Savin, the idea is to parameterize morphisms of rings R into the corresponding cubic norm structure for H. As an example over a number field, in the case of $\mathrm{H} = \mathrm{PU}_3$, we consider the cycle $Z(R) = \{[(A, \iota, \lambda), \phi : R \to \mathrm{End}(A)]\}$ parameterized 3-dimensional abelian schemes A with CM equipped with an action of the cubic ring R, which maps to the Shimura variety for $U_{2,1}$. This appears to give a family of zero cycles, and we suspect that the generating series is related to a modular form on G_2. We plan to continue working on this and other examples involving exceptional dual pairs.

Arithmetic CM cycles on GSp.
$_4$ *and modularity*

Let $\mathcal{A}_2$ be the moduli space of principally polarixed abelian schemes (more generally one could work with $\mathcal{A}_n$ or $\mathcal{A}_1 \times \mathcal{A}_1$). Fix a prime p. For a positive integer n, let $\mathcal{C}(n)$ be the moduli stack classifying triples (A, λ, x) in which $(A, \lambda) \in \mathcal{A}_2$, and $x \in \mathrm{End}(A)$ is an endomorphism satisfying $x^* = -x$, $\mathrm{Tr}(xx^*) = 2n$, and having charactristic polynomial irreducible modulo p. The last condition assures that $\mathcal{C}(n)(\mathbb{C})$ is a finite set. We can prove that the generating series

$$\sum_{n \geq 0} \mathrm{Deg}(\mathcal{C}(n)(\mathbb{C})) \cdot q^n \in \mathbb{Q}[[q]]$$

is a modular form of weight 5.

Let $g_{GS}(n, v)$ be the Kudla and Garcia-Sankaran Green current for the 0-cycle $C(n) = \mathcal{C}(n)(\mathbb{C})$ on $\mathcal{A}_2(\mathbb{C})$, and form the arithmetic cycle

$$\hat{\mathcal{C}}(n, v) = (\mathcal{C}(n), g_{GS}(n, v)) \in \widehat{\mathrm{CH}}^3(\mathcal{A}_2).$$

The Green currents and arithmetic cycles can be extended to all integers ($n = 0$ needs special attention as usual). A fundamental question, which might be very hard, is whether

$$\Theta^{ar}(\tau) = \sum_{n \in \mathbb{Z}} \widehat{\mathcal{C}}(n, v) q^n \tag{0.0.1}$$

is a modular form of weight 5 valued in $\widehat{\mathrm{CH}}^3(\mathcal{A}_2)$.

A more practical question is to prove that $\theta^{ar}(\tau)\cdot\widehat{\omega}$ is modular and identifies it somehow. Here $\widehat{\omega}$ is a metrized Hodge bundle.

Remark 0.0.1. Replacing $\mathcal{A}_2$ by $\mathcal{A}_1$, the above problem might be related to the work of Kudla-Rapoport-Yang in 2024 Compositio Math. The case of $\mathcal{A}_1 \times \mathcal{A}_1$ or Hilbert modular surfaces should be already very interesting.

Remark 0.0.2. Mihatsch, Sankaran, and Yang are working on similar questions for $U(n,1)$ Shimura varieties. One of the main ideas—arithmetic relative trace formula is not available here.

Local modularity.

The goal of this project was to give a more geometric formulation of modularity in a local mixed characteristic setting, in the vein of the conjectures (and theorems) of Feng-Yun-Zhang over function fields. The hope was that this would lead to a more conceptual understanding of certain numerical results that play a crucial role in the proofs of Li and Zhang of the local Kudla-Rapoport conjecture.

The geometric space here was a basic Rapoport-Zink space $\mathcal{N}$ of Kudla-Rapoport type, associated with a p-divisible group with multiplication by the ring of integers in a unramified quadratic extension k of $\mathbb{Q}_p$. Its Tate module gives a $\mathbb{Q}_p$-vector space V_p.

One would then like to define a functional from the space $\mathcal{S}(V_p)^{K_p}$ of spherical vectors in the Schwarz space of $\mathbb{C}$-valued functions on V_p to a certain space of 'cycles' living over the product $\mathbb{V}\times\mathcal{N}$, where $\mathbb{V}$ is the 'nearby' Hermitian space related to V_p via Dieudonné theory and p-adic Hodge theory. This space should admit an action of $\mathrm{SL}_2(\mathbb{Q}_p)$. 'Local modularity' would then amount to the statement that this functional intertwines this action with the Weil representation.

The discussions indicated that, while K-theory with proper supports might be the right target for the non-singular classes, one needed something more refined for the singular ones.

We computed some examples when $\dim \leq 5$, and discussed the use of the methods of Feng-Yun-Zhang applying towards a *proof* of the mixed characeristic formulation, as well as potential applications of local modularity to a new proof of global modularity via p-adic geometry.

Higher weight generating series.

The initial motivation for the project was to construct an analogue of the higher weight Gross-Zagier formula (due to S. Zhang) in the setting of unitary Shimura varieties. While the precise construction of unitary variants of Heegner cycles is not quite clear, they should have cohomological incarnations as classes in the cohomology of the Shimura variety with coefficients in local systems. As a first step towards understanding these cycles, we reviewed work of Funke and Millson that incorporates coefficient systems in the constructions of Kudla-Millson Schwartz forms; significant effort was spent understanding the construction in the Fock model. The hope is to first reconstruct that closed differential forms Funke-Millson using Quillenâ€™s theory of superconnections, so as to construct solutions to the Green equation for these forms.

$\mathbf{U}(\mathbf{a},\mathbf{b})$.

This groups recalled the moduli problem represented by Shimura varieties for the groups $U(a, b)$, and observed that they contain two classes of special cycles: some $Z(n)$ of codimension a, and some $Z^{-}(n)$ of codimension b. We noted that even in the case $a = b$, these two are subtly different as cycles. We considered several generating series built out of these special cycles:

(1) $Z(1)$ intersected with the generating series sum $Z^{-}(n)q^{n}$.
(2) $Z^{-}(1)$ intersected with the generating series sum $Z(n)q^{n}$.
(3) The product of the two generating series: $\sum Z(n)Z^{-}(m)q_1^{n}q_2^{m}$.

We expect that all of these series are modular with values in the arithmetic Chow group, and we worked on verifying this in the simple case where $a = b = 1$, where the arithmetic intersection is supported on the supersingular locus.