Invariant descriptive computability theory

organized by
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Workshop Summary

In our workshop, we brought together computability theorists and set theorists, each of which were working on problems about classification of the complexity of equivalence relations. This workshop had an inherently interdisciplinary perspective and each of our 5 main working groups had members from each discipline. What follows is a report from several of our working groups:

Effectivizing Jump Operators.

In the context of Borel equivalence relations, several jump operators have been thoroughly examined, including the Friedman-Stanley jump and the Louveau jump. Moving to countable equivalence relations, finitary and computable analogs of the Friedman-Stanley jump have been recently explored. Inspired by this earlier work, and with the goal of deepening the natural connection between Borel and computable reducibility, this group initiated a systematic investigation of how to effectivize many other jumps considered by descriptive set theorists.

We first concentrated on the class of Bernoulli jumps, introduced by Clemens and Coskey for Borel equivalence relations. These jumps are associated with countable groups Γ in the following way: The Γ-*jump* of an equivalence relation E is the equivalence relation $E^{[\Gamma]}$ on X^Γ by

$$x\, E^{[\Gamma]}\, y \Leftrightarrow (\exists \gamma \in \Gamma)(\forall \alpha \in \Gamma)(x(\gamma^{-1}\alpha)\, E\, y(\alpha)).$$

We suggested an effectivization of this notion which is natural and results in equivalence relations defined on indices of computable functions. (In fact, it turns out that there are really a few such definitions, depending on how one chooses to handle cases in which computable functions diverge. This issue applies as well to the computable Louveau jump discussed below).

Denote the identity of natural numbers by $\mathrm{Id}(\omega)$. As a case study, we examined the behaviour of the Bernoulli jump of $\mathrm{Id}(\omega)$ when the underlying group Γ changes. We obtained the following dichotomy result:

(1) If Γ is finite, the Bernoulli jump of $\mathrm{Id}(\omega)$ is a simple as possible (being computably bi-reducible with the identity of c.e. sets);
(2) If Γ is infinite, the Bernoulli jump of $\mathrm{Id}(\omega)$ is as complicated as possible (being computably bi-reducible with the eventual equality of c.e. sets).

The proof heavily relies on a similar dichotomy result that Andrews and San Mauro recently obtained when studying orbit equivalence relations induced by suitable actions of computable groups on the space of c.e. sets.

Then, we moved to the effectivization of the class of Louveu jumps. These jumps are associated with filters on ω. In line with the classic notion, for E on ω, we defined its *computable Louveau jump* $E^{\mathcal{F}}$ with respect to a filter $\mathcal{F}$ on ω by

$$x\, E^{\mathcal{F}}\, y \Leftrightarrow \{n : \phi_x(n)\, E\, \phi_y(n)\} \in \mathcal{F}.$$

Once more, we focused on comparing the Louveau jumps of $\mathrm{Id}(\omega)$ given by various filters. In particular, we analyzed principal filters, i.e., filters of the form $\{X : X \supseteq A\}$ for some set A of natural numbers. While principal filters don't give rise to interesting Louveau jumps in the Borel setting, the situation drastically changes in the effective context. Indeed, we constructed an infinite antichain of degrees realized by Louveau jumps of $\mathrm{Id}(\omega)$ with respect to principal filters generated by co-c.e. sets A. On the other hand, if one requires A to be c.e., than the corresponding collection of Louveau jumps of $\mathrm{Id}(\omega)$ is readily classifiable.

A plethora of questions remain open, many of which we intend to address in the future.

Continuous Reductions.

The aim of our group was to study the structure of continuous reducibility $\leq_c$ on smooth equivalence relations, i.e. equivalence relations defined on some (zero-dimensional) Polish space which are reducible to the identity on the reals $\mathrm{id}(\mathbb{R})$. We obtained a crucial result showing that within such class there is a strictly increasing $\leq_c$-chain of length ω_1 which is $\leq_c$-cofinal in the class itself. This has various interesting consequences, such as the fact that there is no $\leq_c$-complete element in the class. The proof is based on the classical stratification of Borel functions in Baire classes, which raised the natural question on what happens if we restrict ourselves to the class of smooth equivalence relations which are reducible to $\mathrm{id}(\mathbb{R})$ via a Baire class α function, for some fixed $\alpha < \omega_1$. One can observe, for example, that by setting $\alpha = 1$ we obtain a class including all closed smooth equivalence relations. In this direction, we proposed a proof of the fact that the partial order given by the restriction of the Wadge hierarchy to sets in $\mathbf{\Sigma}^0_2 \cup \mathbf{\Pi}^0_2$ can be embedded into the class. This in particular proves the existence of $\leq_c$-chains of length $\omega_1^{\omega_1}$. We also observed that this might scale up to other α's, although details need to be checked.

Equivalence Relations from Model Theory.

The aim of our group was to investigate for which levels of the Borel hierarchy there are classes of countable structures such that the isomorphism relation on these classes is complete as a set. We were able to obtain results for some levels and came up with promising strategies to settle the other cases. Specifically we showed that $\mathbf{\Sigma}^0_1$ and $\mathbf{\Sigma}^0_\lambda$ for λ a limit ordinal can not be the complexity of the isomorphism relation on a class of countable structures. For $\mathbf{\Sigma}^0_2$, $\mathbf{\Delta}^0_1$, $\mathbf{\Pi}^0_1$, $\mathbf{\Pi}^0_2$, $\mathbf{\Delta}^0_2$, $\mathbf{\Delta}^0_3$ we were able to construct classes witnessing the completeness of an isomorphism relation at these levels and we have a proposed proof that allows us to transfer these results to all finite levels n. To proceed through the transfinite we have a strategy that combines techniques from descriptive set theory and computable structure theory. Verifying this strategy will require more work.

We plan to collaborate on this project and on promising follow-up questions in the future.

Computable content of hyperfiniteness.

An equivalence relation E is said to be hyperfinite if there are a sequence of equivalence relations $F_0 \subseteq F_1 \subseteq \dots$ so that each F_i has finite classes and $E = \bigcup_i F_i$. A major long-standing open problem in invariant descriptive set theory asks whether hyperfiniteness can always be effectively witnessed. That is, if E is an equivalence relation which is Δ^1_1 and E is hyperfinite, then must there be a uniformly Δ^1_1 sequence F_i witnessing this. We considered first finite-level versions of this and showed that for any $\alpha < \omega_1^{CK}$ there is a Π^0_1 equivalence relation E so that E is hyperfinite but not in a uniformly Σ^0_α way. The proof uses an old unpublished result of Leo Harrington's.

This group still has its eyes on the big prize of the classic open problem, and we are considering several effective variants which give ways to approach it. The group continues to meet regularly.