Combinatorics and complexity of Kronecker coefficients

organized by
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Workshop Summary

The *Kronecker coefficients* $g(\lambda, \mu, \nu)$ are perhaps the most challenging, deep and mysterious objects in Algebraic Combinatorics. Universally admired, they are yet to be understood. For decades since they were introduced by Murnaghan in 1938, the field lacked tools to study them, so they remained largely out of reach. However, in recent years a flurry of activity led to significant advances, spurred in part by the increased interest and applications to Representation Theory, Quantum Computing and Complexity Theory via a program designed to prove the "P vs NP" Millennium problem.

One of the major open problems in the representation theory of the symmetric group in characteristic 0 is that of deciding if there is a combinatorial or geometric description of the Kronecker coefficients, either by giving such a description or by showing that this is not possible. In particular, this leads to the problem of *positivity of Kronecker coefficients*, to determine whether $g(\lambda, \mu, \nu) > 0$.

The workshop had a number of activities organized around the positivity problem. There were talks by Christine Bessenrodt and Igor Pak, continued by discussions led by Rosa Orellana, Greta Panova and Ernesto Vallejo, reporting on various aspects of the *Saxl conjecture* which states that

$$(*) \qquad g(\rho_k, \rho_k, \lambda) > 0$$

for all $\lambda \vdash k(k+1)/2$ and $\rho_k = (k, k-1, \ldots, 1)$. Specifically, the Saxl conjecture has been known at this time for various families of λ such as hooks, two-row partitions, etc. Vallejo explained how to apply his diagrammatic method to prove that, when we fix a partition $\overline{\nu}$ of d and let k vary, the Kronecker coefficient $g(\rho_k, \rho_k, (|\rho_k| - d, \overline{\nu}))$ is a piecewise polynomial function of k. Bessenrodt suggested that for general (non-triangular) partition size n the role of ρ_k should be played by self-conjugate p-cores which are known to exist for all sufficiently large n. The idea is based on integrality of characters of the symmetric groups which gives a remarkable connection to projective and modular representations. Thus, in principle, one can derive positivity of Kronecker coefficients via properties of decomposition matrices.

We also heard a discussion by Christian Ikenmeyer of his most recent breakthrough proving the Saxl conjecture for $\lambda \succ \rho_k$ in dominance order. Finally, Amitai Regev presented a variation on $(*)$ for partitions with few rows which comes from the area of *polynomial identities*. Namely, for every $k > 1$, define $m_k(\lambda)$ as follows:

$$\bigoplus_{\ell(\mu) \le k, \mu \vdash n} \chi^\mu \otimes \chi^\mu = \bigoplus_{\lambda:\, \ell(\lambda) \le k^2, \lambda \vdash n} (\chi^\lambda)^{\oplus m_k(\lambda)},$$

Then $m_k(\lambda) > 0$ for all $\lambda \vdash n$ with $\ell(\lambda) \le k^2$. In a special case $n = k(k+1)/2$ this is easily implied by the Saxl conjecture.

Also during the discussion sessions, a group led by Bessenrodt aimed at resolving a 15 year old problem of classifying the multiplicity free Kronecker products of characters of the symmetric groups. The participants made significant advances based on Manivel's extension of the *semigroup property*:

$$g(\lambda+\alpha,\mu+\beta,\nu+\gamma) \geq g(\lambda,\mu,\nu) \quad \text{whenever} \quad g(\alpha,\beta,\gamma) > 0.$$

The participants expressed confidence that the problem will be completely resolved within a few months of more.

Another important subject of discussion was the stability property, which first appeared in the work of Murnaghan without proof, and later was given several proofs, for example by Littlewood (1958), Brion (1993), Thibon (1991), Vallejo (1999), Briand-Orellana-Rosas (2011). The property states that, for n big enough, the Kronecker coefficient remains constant if we increase the first parts of λ, μ and ν. We heard the reports by Emmanuel Briand, Laura Colmenarejo, Laurent Manivel and Ernesto Vallejo on different aspects, various results and approaches to the problem. Specifically, Vallejo derived the stability from his recent diagrammatic method, Manivel explained how a more general kind of stability, called additive stability, can be derived from the Borel-Weil theory, while Vallejo explained how to derive it using ideas from discrete tomorgraphy. Besides Briand and Colmenarejo gave combinatorial proofs of the stability of some *plethystic coefficients*, an important variation on Kronecker coefficients.

One of the central themes of the workshop was the role played by Kronecker coefficients in the *Geometric Complexity Theory*, the field established by Mulmuley and Sohoni aimed at proving the "P vs NP" problem. Specifically, let the input be integers N, ℓ, partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, $\nu = (\nu_1, \ldots, \nu_\ell)$, where $0 \leq \lambda_i, \mu_i, \nu_i \leq N$, and $|\lambda| = |\mu| = |\nu|$. As part of the program Mulmuley conjectures that the positivity decision problem on whether $g(\lambda,\mu,\nu) > 0$, is in P, and the computing $g(\lambda,\mu,\nu)$, is in #P. These conjectures mimic the known analogous properties of Littlewood–Richardson (LR) coefficients, however the situation is already exacerbated by the fact that the Kronecker coefficients do not share the saturation property of the LR coefficients, i.e. it is not generally true that if $g(N\lambda, N\mu, N\nu) > 0$ then $g(\lambda,\mu,\nu) > 0$.

The workshop showcased lectures by Peter Bürgisser, Christian Ikenmeyer and Joseph Landsberg on the subject, offering different perspectives. Landsberg made very specific questions asking for examples of inequalities between certain (symmetric) Kronecker and plethistic coefficients. Later during discussion and open problem sections, Ikenmeyer gave an explanation of the fascinating connection with the celebrated *Alon–Tarsi Conjecture* counting Latin squares according to sign. Landsberg also mentioned a different connection with this conjecture which he recently posted on the arXiv.

In another direction, we heard talks by Matthias Christandl and Michael Walter on connections of Kronecker coefficients and Quantum Information Theory related to the spectra of quantum states, following Klyachko (2004), Christandl–Mitchison (2006), Christandl–Harrow–Mitchison (2007). The speakers presented construction of what they call *Kronecker polytopes* which describe positivity of the *stretched Kronecker coefficients*. Walter presented remarkable and highly technical work he did with Michèle Vergne which described the facets of the polytopes.

In the discussion that followed, the group led by Christandl and Walter concluded that it seems the *membership test* for stretched Kronecker coefficients is both in NP and co-NP, suggesting it might be in P. To put this in plain language, this means that for every three rational distributions $\alpha_1+\ldots+\alpha_\ell=1$, $\beta_1+\ldots+\beta_\ell=1$, $\gamma_1+\ldots+\gamma_\ell=1$, $\alpha_1\geq\ldots\geq\alpha_\ell\geq 0$, $\beta_1\geq\ldots\geq\beta_\ell\geq 0$, $\gamma_1\geq\ldots\geq\gamma_\ell\geq 0$, one can perhaps efficiently decide if there is an integer k such that $g(k\alpha,k\beta,k\gamma)>0$ whenever these partitions are well defined.

We also discussed various combinatorial properties of Kronecker coefficients. In particular, Jeff Remmel led a discussion group on his B_n-analogue of Kronecker product of symmetric functions, which remain difficult to understand. Stephanie van Willigenburg gave a talk and led a discussion on the analogues of the Kronecker product for *quasi-symmetric functions*. Rosa Orellana and Chris Bowman explained how one can obtain the *reduced Kronecker coefficients* $\overline{g}(\lambda,\mu,\nu)$ in terms of *partition algebra* representations, perhaps suggesting that $\overline{g}(\lambda,\mu,\nu)$ might have a combinatorial interpretation from another direction.

Jonah Blasiak and Ricky Liu explained their new version in terms of integer points in polytopes of Blasiak's combinatorial interpretation of Kronecker coefficients when one partition is hook. This is in stark contrast with the case of two-row partitions where only partial results are known, cf. Ballantine–Orellana (2005, 2007), Blasiak–Mulmuley–Sohoni (2012), Lascoux (1980), Remmel (1989), Rosas (2001), Brown–van Willigenburg–Zabrocki (2010) and Pak-Panova (2013, 2014).

Finally, a group led by Alexander Yong and Steven Sam discussed the possible generalization of Kronecker coefficients in the context of K-theory. In particular, the definition of Kronecker product on the space of symmetric functions could be extended through Schubert polynomials to the space of all polynomials.