Geometry of large networks

organized by
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Workshop Summary

The workshop aimed at connecting several rather diverse groups of mathematicians, engineers, computer scientists and physicists trying to model the large networks from a more geometric perspective.

The primary goal of the workshop was to create a common language, and basic list of problems among these constituencies mathematicians dealing with large scale "networks" (anything, in essence, that can be used to model networks, such as infinite graphs and harmonic analysis on them, geometric group theorists, l_2-topologists, probabilists dealing with random walks), computer scientists, especially those specializing in large-scale structures, physicists (who have a cottage industry of analysing and simulating large networks, using primarily metaphors from statistical physics) and electrical engineers, who provide most of the actual instances of large communication networks.

(It should be noted that the emphasis of the workshop was on traffic phenomena in the networks, essentially focussing the discussion on the communication networks, and leaving aside many of other application areas, such as "social networks", various biological networks etc.) Looking back, it is clear that mathematics was a little bit underrepresented, as well as some of the CS communities.

The workshop started with two introductory lectures: Francis Bonahon gave an overview of hyperbolic geometry, with a view towards properties that might find applications in modeling communication networks; Dmitri Krioukov in turns delivered a talk on his results where he explicitly used (real) hyperbolic space as a model for "Internet", and showed how this could be used in solving routing problems.

These two talks set the background for the group discussion that happened on the first day, where the participants attempted to create a glossary of terms, notions and memes used by different constituencies. Even the notion of hyperbolic space turned out to be used differently by different groups. Several notions, like that of "ensemble" were disambiguated, inasmuch as the circumstances permitted. That discussion led to creation of a list of problems and a short working period by the breakout groups.

The next day started with Iraj Saniee describing his and collaborators efforts in quantifying "hyperbolicity" of certain networks, both simulated and real. This was a important talk for most non-engineers in the audience, as it clearly showed where the emphasis of practitioners lies, and also how the non-positive curvature could manifest itself in telecommunication networks.

Shmuel Weinberger then gave a talk a collection of topological characteristics of large-scale metric spaces. In particular, the question of dimension, asymptotic boundary and Gromov hyperbolicity were discussed.

In the afternoon, the working groups formed in the earnest, and serious attempts started to address several of the pressing questions raised during the first day. The largest group formed around the quintessential question on what is hyperbolicity (or, more generally, curvature) in large networks. In many respects this group was continuing the afternoon discussion of the Day 1, serving primarily the purpose of settling on a common set of terms, definitions and notations. On the second day this, initially the largest, group eventually saw migration of its members to more focussed working groups which worked on specific questions.

These questions included:

- **Congestion** How to measure congestion in networks? There are several common ways it is defined in the practice of telecommunications, but they all include a specific resource (link, or frequency band), and to not lend readily to the geometric viewpoint that was driving the workshop.
- **Finite-to-global** The Δ-hyperbolicity of the networks is (conceptually) easy to verify when the network is sufficiently symmetric (say, is a geodesic subset of a graph (CW complex) that admits a co-compact action), but would require to check an awful lot of inequalities in general. The natural question is whether one can effectively produce a (finite) collection of conditions that would ensure that the network is hyperbolic? Perhaps by relaxing the hyperbolicity "at all scales" one still could arrive at (somewhat weaker) results: say, can the exponential growth be deduced from the Δ-hyperbolicity at a finite scale and some constraints on the lengths of geodesic cycles?
- **Ensembles and Exponential families** One of the pervasive tools in analysis and generation of large network is the exponential family, or, equivalently, Gibbs ensemble, biased by some characteristic ("fugacities, clustering,...") of the network that one aims to understand. The question whether the Gromov hyperbolicity of a network can be represented in such a way might be vital for understanding and simulation of large-scale hyperbolic networks. While there are certainly some (tautological) approaches leading, formally, to the families of hyperbolic networks, it is not clear how to use them effectively.
- **Spectral properties** Large networks has been traditionally a fertile playground for spectral theory. The spectral properties of the large-scale networks are relatively difficult to control from the geometric data, yet this is one of the most powerful tools in harmonic analysis, especially in homogeneous situations.
- **Distance between graphs** Considering the finite metric spaces (or their inductive limits) as points of the space of metric spaces with Hausdorff norm on them is one of the most fruitful ideas in Gromov hyperbolicity. How to work with these set of ideas in finite (but large) setting? What are the metric (and topological) properties of the subsets of metrics with bounds on the curvature?

These groups covered a lot of ground in setting up their research agendas, and in bringing together the folks from different areas. With some variations, there groups continued working throughout the workshop, gathering in the afternoon, with morning dedicated to survey talks and reports on the progress.

Wednesday morning started with two survey talks - one by Palle Jorgensen, who gave an overview of the spectral theory of infinite networks, with an emphasis on their boundaries, and the other by Susan Holmes who was talking about tree spaces and their applications in inference problems. Susan's approach, eminently finitary, was seen as an invitation to look at the space of all (finite) metric spaces, and to investigate the subsets of Δ-hyperbolic spaces, filtered by Δ.

On Thursday Michael Mahoney gave an extremely interesting talk on the structures of large networks coming from several large databases he and his collaborators were studying. Most of them were not directly related to telecommunications, but the tools and the viewpoint Michael advertised should be playing an important role in any study of "real-life" networked structures. In essence, his message was that the networks are very messy, with heterogeneous pieces defying any specific model, but these pieces in a sense repeat themselves at different scales. While very hard to formalize, this phenomenological approach was certainly perceived by the participants as very appealing and lead to a heated discussion.

Ginestra Bianconi talked about the the complex networks from statistical physics perspective. The discussion returned to the notion of the exponential families and their relevance for the hyperbolic networks, and, more generally, to the geometric view of the large-scale structures.

On Friday, Jie Gao talked about the networks generated (and simplified) via conformal transformations. This approach was manifestly geometric, but the discussion did not lead to a consensus on how well such approach describes realistic networks. It seemed that the powerful machinery of the uniformization is taken over the concerns over its applicability.

The last formal talk of the workshop was delivered by Blair Sullivan. By that time it became clear that graph-theoretic tools are too rich and powerful to ignore them. Blair's talk exhibited an amazing richness of the results on the structural theory of large graphs, revolving around the Robertson-Seymour minor theory.

To our knowledge, there was and still is quite a bit of continuing interactions between the participants after the workshop. It is still hard to assess what is the exact degree of the cross-fertilization, but some specific projects are alive and, in some instances, approaching a publisheable stage.

A lot of action happened around the question whether or note the real-life networks should be actually modeled by the *two-dimensional* hyperbolic space. Main proponent of this viewpoint, Dima Krioukov was pointing at the numeric evidence, while main skeptic, Shmuel Weinberger was questioning whether these data have any bearing on the dimension. Krioukov and Weinberger are collaborating on this topic, planning to evaluate numeric evidence using topological tests (to estimate the "persistence" dimension of the Cech complex built from the network Dima is studying).

Susan Holmes' quest for understanding the filtration of the space of finite metric spaces by Δ-hyperbolic ones, and its relation to the subset of the ultrametrics resonated with Dan Guralnik and Jesse Johnson. They are working on this.

The question about interplay between graph-theoretical notions and Gromov hyperbolicity was addressed by the Adcock, Baryshnikov, Mahoney and Sullivan. They attacked the connection between the treewidth of a network, and are looking for an extra condition that would allow one to relate it to the hyperbolicity. The results are promising, and a preprint is being in preparation.

All in all, this was an extremely useful workshop, for both sides who were exposed to the modalities of scientific inquiry from across the aisle. One can expect collaborations, but more importantly, engineers, computer scientists and mathematicians who took part in the workshop gained a sort of binocular vision in the topic of geometry of large networks. Explicitly, or implicitly, this exposure is going to manifest itself in much enriched research.