AIM Workshop on Configuration Spaces of Linkages 
Open Problems

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1. (Farber) Configuration spaces of closed polygons in $\mathbb{R}^3$. Let $\ell = (\ell_1, \ldots, \ell_n)$ be the length vector defining a closed polygon.

The configuration space is $M_\ell \backslash \Sigma = \sqcup U_i$, where $U_i$ are the connected components, and $\Sigma$ is the set of configurations with self-intersections.

Questions: Are the $U_i$ contractible? What can we say about their topology?

2. (Streinu) Same as the previous question, but with an open arm in $\mathbb{R}^3$, and $\ell_1 = \ell_2 = \cdots = \ell_n$.

Question: Is the space of non-intersecting configurations connected?

3. (Farber/Panina) Consider a closed polygon in $\mathbb{R}^3$. According to Klyatchko, $M_\ell$ has a symplectic form for $\omega$, and therefore for volume.

$$Vol\left(\sqcup U_i\right) / Vol(M_\ell),$$

where $i$ corresponds to an unknot.

(Note if $\ell_1 + \ell_2 + \cdots + \ell_{n-1} = \ell_n$, this is always an unknot and therefore the proportion above is exactly 1. (see below))

Question: try to understand this proportion.

4. (Holmes-Cerfon) Consider the configuration space of planar $n$-gons such that

$$L_i - \epsilon \leq \ell_i \leq L_i + \epsilon.$$

$M_{L,\epsilon}$ is a manifold with boundary.

Question: Understand its topology and volume.
5. (Sitharam) Consider two polygons that share a “chain” (see below), or graphs of tree-width 2.

Question: Apply Morse Theory à la Farber to derive the Betti numbers:

a) Chambers
b) Homologies

6. (Farber) Question: Find asymptotic behaviour of $C_n$ (the number of orbits of chambers in the case of polygonal linkages).

7. (Thorpe) Consider a network of corner-sharing triangles in the plane, with holes of size 5, 6, 7, 8, 9 (and an average hole size of 6). If this is an infinite network, it is isostatic.
Now take a large finite piece of the framework. Experimentally, if we pin every other triangle boundary vertex (vertex of degree two) and run the pebble game, we get an isostatic network.

Question: prove that this approach works in general and find other distributions of pins that also work.

a) generic
b) equilateral triangles

Question (generalization): What about the class of graphs that have no proper rigid subgraphs, but that have more than two bodies at a pin? The underlying body-pin graph is no longer 3-regular.

8. (Hempel) Consider a simplicial polyhedra with fixed combinatorics in $\mathbb{R}^3$.

Question: characterize those collections of dihedral angles and face angles that can be realized by a polyhedron with these combinatorics. Conjecture: dimension $= E - 1$, where $E$ is the number of edges. Needs more clarification.

9. (St. John) Consider a multi-robot formation that is a generically minimally rigid framework $G$, with diameter $D = \max_{(p_i, p_j)} ||p_i - p_j||$. We remove an edge and obtain $\bar{G}$ that is flexible. Let $d = \min \text{diameter}(\text{configuration space of } \bar{G})$.

Question: Understand the relationship between $D$ and $d$, and find algorithms to detect what edge to delete for maximal change in diameter. More precisely, find bar $e$ and “positioning” $q$ such that the diameter of $(G \setminus e, q)$ is minimum under the constraints that bar lengths in $G \setminus e$ are maintained.

10. (Theran) Consider a Delaunay triangulation (no vertex is inside the circumcircle of any triangle). Fix a combinatorial type of a triangulation.

Question: What is the configuration space?

$d = 2$ it is a ball

$d = 3$ is it universal?

11. (Owen) Specialization: When is the Galois group of a particular rigid framework a subgroup of the Galois group of the graph?

When is $\mathcal{G}(G, p) \subseteq \mathcal{G}(G)$ for any $p$ with $(G, p)$ isostatic? (Isostatic is sufficient but not necessary).

Here $p$ generic means $p = x_1, \ldots, x_n$ are algebraically independent over $\mathbb{Q}$.

12. (Whiteley) Conjecture: Given a symmetric bar-joint framework $(G, p)$, the configuration space of $(G, p)$ (with appropriate parts of the frame of reference fixed), has the symmetry of the most symmetric individual realization in the configuration space.
13. (Schulze) Understand the following question: does the pseudo triangulation algorithm for the Carpenter’s Rule give some unfolding that preserves symmetries? (Note that Connelly, Demaine, Rote have non-algorithmic positive solution).

14. (Schulze) Suppose a symmetric framework (linkage) has a 1DOF expansive mechanism. Does the mechanism preserve the symmetry?

15. (Schulze) Under what conditions is a linkage guaranteed to preserve the original symmetry throughout the configuration space?

16. (St. John/Schulze) Persistence theory: Group of connected agents, every agent has out-degree 2 (except for 2 agents, the leader and the co-leader).

Question: Can symmetry of the configuration be exploited to reduce the computation? What about a body-CAD version?