# PROBLEMS ON THURSTON METRIC

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This is a list of problems, with short comments, related to the study of the Thurston metric on Teichmüller space.

The problems summarize the discussion by the participants in the AIM workshop with title "Lipschitz metric on Teichmüller space". Some of the problems have been worked on by small groups and they remain open, and some other problems have been suggested by various participants at the closing problem session of the conference. I apologize for missing some of the problems.

In the following, we denote by $S$ a hyperbolic surface of genus $g$ with $n$ punctures and by $\mathcal{T}(S)$ the Teichmüller space of $S$.

## 1. Infinitesimal properties of the Thurston metric

**Problem 1.1** (D. Dumas, K. Rafi)**.** Does the Thurston norm at $X \in \mathcal{T}(S)$ determines $X$?

Problem 1.1 is motivated by Royden's proof of his celebrated theorem that (roughly speaking) the isometry group of Teichmüller space endowed with the Teichmüller metric is the mapping class group.

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**Problem 1.2** (K. Rafi)**.** What properties of the hyperbolic surface are determined by the Thurston infinitesimal norm?

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**Problem 1.3** (F. Guértitaud)**.** Does the unit sphere in $\mathrm{T}^*_X\mathcal{T}(S)$ endowed with the Thurston norm determine the global behaviour of stretch lines or geodesics.

As shown by Thurston, the behavior of lengths of laminations infinitesimally in measured lamination space gives good global coordinates for Teichmüller space.

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**Problem 1.4.** Is each local isometry of Teichmüller space with the Thurston metric induced by an element of the extended mapping class group?

It was recently observed by Walsh that the horofunction compactification of Teichmüller space with the Thurston metric is naturally identified with Thurston's compactification.

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**Problem 1.5** (D. Dumas)**.** Is there any sense in which the Thurston metric is compatible with the complex structure of Teichmüller space?

We may ask the same question for symplectic structure.

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## 2. Geodesics of Thurston metric

Any two points in Teichmüller space can be joined by a geodesic path that is a concatenation of stretch segments.

**Problem 2.1.** Is there an algorithm to find the maximally stretch lamination?

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Generically, a Thurston geodesic connecting two points in Teichmüller space is not unique.

**Problem 2.2** (F. Guéritau)**.** Given $X, Y \in \mathcal{T}(S)$, describe the set

$$\mathrm{Env}(X, Y) := \cup\{\mathcal{G}\},$$

where $\mathcal{G}$ denotes a Thurston geodesic connecting $X$ to $Y$.

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**Problem 2.3** (K. Rafi)**.** Does $\mathrm{Env}(X, Y)$ depend continuously on $X, Y$?

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**Problem 2.4** (K. Rafi)**.** Identify curves that are short along the preferred path $\mathcal{G}_{X,Y}$.

F. Kassel and K. Rafi prove that, for any $X, Y \in \mathcal{T}(S)$, there is a Thurston geodesic $\mathcal{G}_{X,Y}$, parametrized linearly in Thurston's shear (cataclysm) coordinates associated with a canonical lamination $\lambda(X, Y)$ such that lengths of all simple closed curves along $\mathcal{G}_{X,Y}$ are convex functions (up to reparametrization).

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**Problem 2.5** (K. Rafi)**.** Assume that $\mathcal{G}(\lambda)$ is a stretch line directed by a maximal geodesic lamination $\lambda$, then how does the geodesic $\mathcal{G}(\lambda)$ look like? Moreover, if the projection of $\mathcal{G}(\lambda)$ to the arc and curve graph of a subsurface $Y$ is large, can we expect that an interval of time where each boundary component $\beta$ of $Y$ has bounded length and $\beta$ is close to a geodesic on $Y$ ?

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**Problem 2.6** (K. Rafi)**.** Can we understand the behavior of Thurston's geodesics inductively, going from surfaces to subsurfaces?

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There have also been a great deal of study on the dynamical properties of the Teichmüller geodesic flow on moduli space or tangent bundle of muduli space, with application to billiards and flat structures.

**Problem 2.7** (K. Rafi)**.** Define the geodesic flow for the Thurston metric and study its properties, such as ergodic or mixing.

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Inspired by the work of Eskin and Mirzakhani on Teichmüller metric, we ask

**Problem 2.8** (K. Rafi)**.** Find the asymptotic behavior for the number of conjugacy classes of pseudo-Anosov elements of the mapping class group whose translation length in the Thurston metric is less than $R$.

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**Problem 2.9.** Is there a dense Thurston geodesic in moduli space?

Masur used closed Teichmüller geodesics to approximate a Teichmüller geodesic which is dense in moduli space.

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**Problem 2.10** (K. Rafi)**.** Is a stretch line typically recurrent/ dense/ e-quidistributed in moduli space?

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The question of when two Teichmüller geodesic rays stay bounded distance apart has been answered completely by Lenzhen and Masur.

**Problem 2.11.** Determine when two Thurston geodesic rays stay bounded distance apart.

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**Problem 2.12** (K. Rafi)**.** Given two points in Teichmüller space, do we have a comparison between the maximal stretch laminations (which define the concatenation of stretch segments) and the vertical (horizontal) foliations of quadratic differential for the Teichmüller geodesic connecting them?

Minsky has studied the approximate behavior of high-energy harmonic maps when the domain surface is varied, and compared the harmonic maps (which are stretched along the Hopf foliation) with Teichmüller maps and stretch maps.

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**Problem 2.13.** What is the quasi-isometric group of Teichmüller space equipped with the Thurston metric?

## 3. Symmetrization of the Thurston metric

**Problem 3.1** (A. Papadopoulos)**.** What is a good way to symmetrize the Thurston metric?

The *length-spectrum metric* by Sorvali is a symmetrization of the Thurston metric.

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**Problem 3.2** (A. Papadopoulos)**.** Is the length-spectrum metric Finsler? If yes, give a formula for the infinitesimal norm of a vector on Teichmüller space with respect to this Finsler structure.

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**Problem 3.3** (A. Papadopoulos)**.** Is $\mathrm{Isom}(\mathcal{T}(S), d_{\mathrm{ls}}) = \mathrm{Mod}(S)$?

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**Problem 3.4** (K. Rafi)**.** Is the Thurston metric a degeneration of some better vector-valued symmetric distance on Teichmüller space?

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The *reversed Thurston metric* $d^*$ is defined by

$$d^*(X,Y) = d_{\mathrm{Th}}(Y,X).$$

**Problem 3.5** (C. Walsh)**.** What is the horofunction boundary of the reversed Thurston metric?

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Let $S$ be a surface of finite type with nonempty boundary. The *reduced Teichmüller space* $\mathcal{T}(S)$ is the set of equivalence classes of marked bordered hypebolic structures on $S$.

**Problem 3.6** (F. Guéritaud)**.** Describe the cone of directions in the tangent space $\mathrm{T}_X\mathcal{T}(S)$ which shorten the lengths of all simple closed curves on $S$.

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The arc metric $d_A$ on $\mathcal{T}(S)$ is a natural generalization of the Thurston metric By doubling, there is a natural isometric embedding from $(\mathcal{T}(S), d_A)$ to $(\mathcal{T}(S^d), d_{\mathrm{Th}})$. We shall identify $\mathcal{T}(S)$ with its image in $\mathcal{T}(S^d)$.

**Problem 3.7.** Is the metric space $(\mathcal{T}(S), d_A)$ a length space? Given $X, Y \in \mathcal{T}(S)$, is there a geodesic $\Gamma$ of $(\mathcal{T}(S^d), d_{\mathrm{Th}})$ connecting $X$ and $Y$ such that $\Gamma \subset \mathcal{T}(S)$?

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**Problem 3.8.** Let $S$ be a hyperbolic surface with ideal boundary. Can we define the Thurston metric on the non-reduced Teichmüller space of $S$?

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**Problem 3.9** (D. Dumas)**.** Can we define the Thurston metric on the universal Teichmüller space?

## 4. Infinitely-generated Fuchsian group of the first kind

Let $\Gamma_0$ be an infinite-generated Fuchsian group of the first kind.

**Problem 4.1.** Can we deform a infinite-generated Fuchsian group of the first kind (or, a closed hyperbolic surface of infinite area) by quasiconformal maps such that the lengths of all simple closed curves are not increased?

Alessandrini and Guéritaud have recent results on this question.

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Problem 4.1 is related to

**Problem 4.2.** Can we defining Thurston metric on $\mathcal{T}_{\mathrm{qc}}(\Gamma_0)$?

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**Problem 4.3** (M. Kapovich)**.** Can we construct example of $\Gamma_0$ (infinite-generated and of the first kind) where the critical exponent of some elements in $\mathcal{T}_{\mathrm{qc}}(\Gamma_0)$ are distinct?

M. Kapovich suggested that the above question maybe related to Problem 4.1. He guessed that the construction depends on the ergodic properties of $\Gamma_0$.

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One idea to understand surfaces of infinite type is approximate it by subsurfaces of finite type. Let $\Gamma_0$ be a infinite-generated Fuchsian group of the first kind. Denote by $\Gamma_n$ the fundamental group carried by the $n$-ball in $\mathbb{H}/\Gamma_0$.

**Problem 4.4** (F. Guértitaud)**.** Does $\delta(\Gamma_n) \to \delta(\Gamma_0)$ as $n \to \infty$?

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**Problem 4.5** (F. Guértitaud)**.** Can we have a quasiconformal deformation of $\Gamma_0$ which bend a variation of $\delta(\Gamma_n)$ independent of $n$?

## 5. Other questions

**Problem 5.1.** What is Thurston metric on $\mathbb{H}^2$, viewed as the Teichmüller space of punctured tori?

Belkhirat, Papadopoulos and Troyanov have studied the Thurston metric on Teichmüller space of flat tori.

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**Problem 5.2** (D. Dumas)**.** Does Thurston's metric have any relation with the following unsolved problem: *Consider all conformal metrics $\rho$ on a Riemann surface which satisfy*

$$\inf_{\gamma} \ell_\rho(\gamma) \geq 1.$$

*Is there always one of such metrics with least area?*

See the work of Wolf and Zwiebach for study of the above question and application to string theory.

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**Problem 5.3.** Define and study the Thurston metric on the space of flat $n$-tori $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$.

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**Problem 5.4** (D. Dumas)**.** How does the "Lipschitz constant function"

$$(g, h) \mapsto \inf_{\phi: g \to h} \mathrm{Lip}(\phi)$$

behave on a class of metrics larger than the set of hyperbolic metrics, for example, the set of negatively curved Riemannian metrics or its closure?