PROBLEMS ON THURSTON METRIC

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This is a list of problems, with short comments, related to the study of the Thurston metric on Teichmüller space.

The problems summarize the discussion by the participants in the AIM workshop with title "Lipschitz metric on Teichmüller space". Some of the problems have been worked on by small groups and they remain open, and some other problems have been suggested by various participants at the closing problem session of the conference. I apologize for missing some of the problems.

In the following, we denote by S a hyperbolic surface of genus g with n punctures and by $\mathcal{T}(S)$ the Teichmüller space of S.

1. Infinitesimal properties of the Thurston metric

Problem 1.1 (D. Dumas, K. Rafi). Does the Thurston norm at $X \in \mathcal{T}(S)$ determines X?

Problem 1.1 is motivated by Royden's proof of his celebrated theorem that (roughly speaking) the isometry group of Teichmüller space endowed with the Teichmüller metric is the mapping class group.

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Problem 1.2 (K. Rafi). What properties of the hyperbolic surface are determined by the Thurston infinitesimal norm?

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Problem 1.3 (F. Guértitaud). Does the unit sphere in $T_X^*\mathcal{T}(S)$ endowed with the Thurston norm determine the global behaviour of stretch lines or geodesics.

As shown by Thurston, the behavior of lengths of laminations infinitesimally in measured lamination space gives good global coordinates for Teichmüller space.

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Problem 1.4. Is each local isometry of Teichmüller space with the Thurston metric induced by an element of the extended mapping class group?

It was recently observed by Walsh that the horofunction compactification of Teichmüller space with the Thurston metric is naturally identified with Thurston's compactification.

Problem 1.5 (D. Dumas). Is there any sense in which the Thurston metric is compatible with the complex structure of Teichmüller space?

We may ask the same question for symplectic structure.

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space is not unique.

2. Geodesics of Thurston metric

Any two points in Teichmüller space can be joined by a geodesic path that is a concatenation of stretch segments.

Problem 2.1. Is there an algorithm to find the maximally stretch lamination?

Generically, a Thurston geodesic connecting two points in Teichmüller

Problem 2.2 (F. Guéritau). Given $X, Y \in \mathcal{T}(S)$, describe the set $\operatorname{Env}(X,Y) := \bigcup \{\mathcal{G}\},$

where $\mathcal G$ denotes a Thurston geodesic connecting X to Y.

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Problem 2.3 (K. Rafi). Does Env(X,Y) depend continuously on X,Y?

Problem 2.4 (K. Rafi). Identify curves that are short along the preferred path $\mathcal{G}_{X,Y}$.

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F. Kassel and K. Rafi prove that, for any $X, Y \in \mathcal{T}(S)$, there is a Thurston geodesic $\mathcal{G}_{X,Y}$, parametrized linearly in Thurston's shear (cataclysm) coordinates associated with a canonical lamination $\lambda(X,Y)$ such that lengths of all simple closed curves along $\mathcal{G}_{X,Y}$ are convex functions (up to reparametrization).

Problem 2.5 (K. Rafi). Assume that $\mathcal{G}(\lambda)$ is a stretch line directed by a maximal geodesic lamination λ , then how does the geodesic $\mathcal{G}(\lambda)$ look like? Moreover, if the projection of $\mathcal{G}(\lambda)$ to the arc and curve graph of a subsurface Y is large, can we expect that an interval of time where each boundary component β of Y has bounded length and β is close to a geodesic on Y?

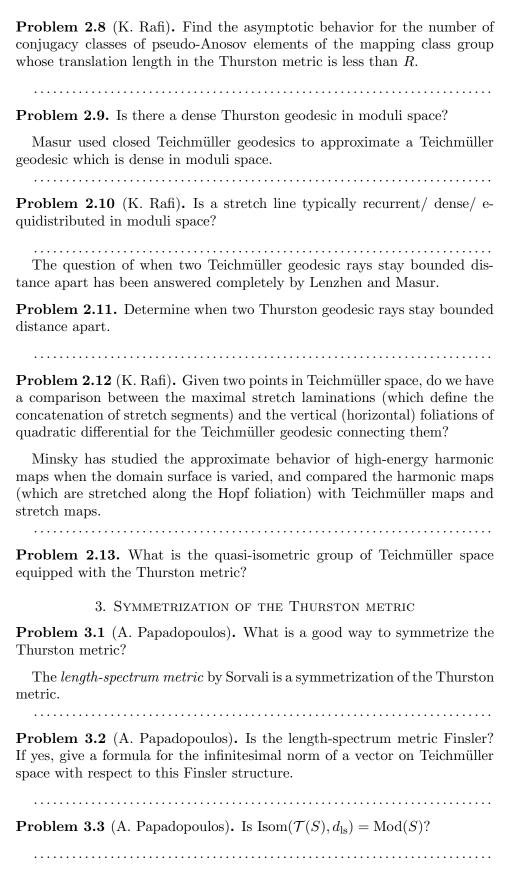
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Problem 2.6 (K. Rafi). Can we understand the behavior of Thurston's geodesics inductively, going from surfaces to subsurfaces?

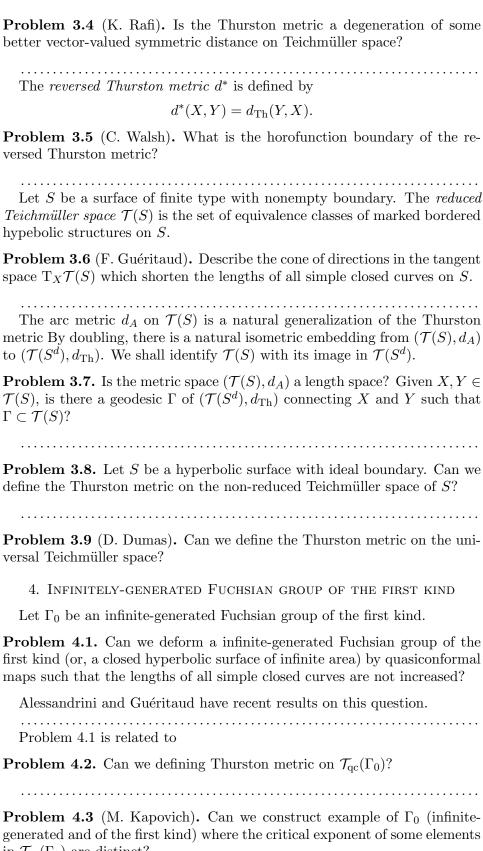
There have also been a great deal of study on the dynamical properties of the Teichmüller geodesic flow on moduli space or tangent bundle of muduli space, with application to billiards and flat structures.

Problem 2.7 (K. Rafi). Define the geodesic flow for the Thurston metric and study its properties, such as ergodic or mixing.

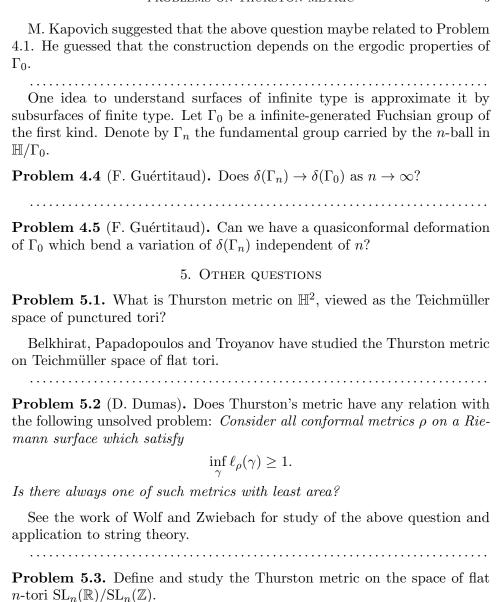
Inspired by the work of Eskin and Mirzakhani on Teichmüller metric, we ask



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in $\mathcal{T}_{qc}(\Gamma_0)$ are distinct?



Problem 5.4 (D. Dumas). How does the "Lipschitz constant function"

$$(g,h) \mapsto \inf_{\phi: g \to h} \operatorname{Lip}(\phi)$$

behave on a class of metrics larger than the set of hyperbolic metrics, for example, the set of negatively curved Riemannian metrics or its closure?