Localization techniques in equivariant cohomology

organized by

William Fulton, Rebecca Goldin, and Julianna Tymoczko

Workshop Summary

The conference attendees consisted of mathematicians from different areas of mathematics who work on localization techniques in equivariant cohomology. The fields represented at the conference included algebraic topology, symplectic geometry, combinatorics, and algebraic geometry; all of these fields have contributed significantly to the study of equivariant cohomology, but often without understanding of related work in other fields. Our goal was to open communication between mathematicians in different fields, to initiate deeper mathematical work and prompt collaborations across disciplines. As such, an important goal was to establish a common vocabulary about the definitions, techniques, and even motivating questions that different researchers considered. (Unofficially, we tried to avoid research talks in favor of surveys.)

The first morning had two survey talks. Sara Billey described what localization meant from her perspective as a combinatorist; she focused on the existence and explicit construction of computationally-useful bases, particularly for topological spaces with important combinatorial attributes (like flag varieties, Grassmannians, and Schubert varieties). Megumi Harada gave a symplectic geometer's perspective on the same, introducing symplectic manifolds M with Hamiltonian T-actions, together with the inclusion of fixed points $\iota : M^T \hookrightarrow M$ and the map ι^* that inclusion induces on T-equivariant cohomology. Her talk addressed some conditions under which the induced map $\iota^* : H_T^*(M) \to H_T^*(M^T)$ is injective. These conditions—and the general notion of *equivariant formality*, which has substantively different meaning for different kinds of mathematicians, but is related to the condition that $H_T^*(M)$ be a free module over $!H_T^*(\mathrm{pt})$—became an ongoing theme in the conference.

The first afternoon had a group discussion with several mini-talks. Volker Puppe and Matthias Franz fleshed out ideas about how algebraic localization could be used to identify the image of the map ι^*. Megumi Harada continued her talk with a discussion of ABBV localization (a theorem of Atiyah-Bott-Berligne-Vergne describing how to compute equivariant integrals); Allen Knutson also made some short comments on the same theme. The day ended with a group discussion of open questions.

The second morning had two survey talks. The first, by Tom Braden, described from an algebraic topologist/geometer's perspective the basic constructions that are sometimes called GKM theory. His fundamental object was a complex algebraic variety X with a T-action such that the number of T-fixed points is finite, the number of one-dimensional T-orbits is finite, and X satisfies equivariant formality (especially that $H_T^*(X)$ be free over $H_T^*(\mathrm{pt})$). In this context, he described explicit combinatorial calculations that give the ring $H_T^*(X)$ based on a graph (called the GKM graph, the moment graph, or the labeled one-skeleton) that encodes the fixed points and one-dimensional T-orbits of X. Sue Tolman gave the second talk, and described a symplectic geometer/algebraic topologist's approach to similar questions. She discussed the extent to which the moment graph of a symplectic manifold M determines the manifold, and how to use the graph to get formlas for computing (via computationally convenient bases).

On the second afternoon, we broke into three groups. Tom Braden gave the largest group an introduction to intersection homology, beginning with the general constructions and theory of intersection homology and ending with the analogy of GKM theory for intersection homology. A smaller group discussed specific calculations in GKM theory, doing some examples outside of Lie type A and asking (and then answering) what the natural map $G(k_1, n_1) \times G(k_2, n_2) \to G(k_1 + k_2, n_1 + n_2)$ induced on equivariant cohomology is. The third group worked on the topic *Beyond equivariant formality* and generated provocative (research-level) ideas about how to proceed with GKM-like analysis without the condition of equivariant formality.

The third day began with two algebraic geometers (Allen Knutson and Dave Anderson) describing toric degenerations: what they are (geometrically and algebraically) and how they can be used, particularly in Schubert calculus contexts (namely, to answer questions about the cohomology of a flag variety or Grassmannian with respect to the basis of Schubert classes). The afternoon saw a vigorous introduction to K-theory, with several tag-team speakers and active audience participation (and chalk-grabbing). Speakers discussed the essential differences between T-equivariant cohomology and T-equivariant K-theory (where the former is a module over a polynomial ring, the latter is a module over Laurent polynomials). They also described and gave examples of the analogue of GKM theory for equivariant K-theory.

On the fourth day we turned to Schubert calculus, which studies the cohomology ring (for many different cohomology theories) of a Grassmannian (or other partial flag variety) in terms of a natural geometric basis of *Schubert classes*. Alex Yong gave a combinatorial perspective on Schubert calculus. He described how Schubert polynomials are a combinatorial object that mediates between classical questions in Schubert calculus and geometric questions about the singularities of Schubert varieties. Linda Chen then spoke from an algebraic geometer's perspective, and presented the three main goals of Schubert calculus from her point of view: 1) an explicit presentation of the cohomology ring that is preferably both algebraically and geometrically natural in some sense; 2) the *Giambelli problem* (to write down an arbitrary Schubert class in terms of a nice additive module basis of special Schubert classes); and 3) the *Pieri problem* (to understand the product of a special Schubert class with an arbitrary Schubert class).

In the afternoon, the group generated a table of what the participants know about Schubert-calculus. (This table can be found on the wiki page associated with this conference.) We then broke into groups, the largest of which talked about affine Schubert calculus (how it relates to classical Schubert calculus, and what's known and unknown). Two similarly-small-sized groups focused on more specific questions: finding equivariant Pieri rules and finding new families of topological spaces that satisfy GKM conditions, but are not homogeneous spaces.

On the morning of the fifth day, we had three shorter talks. Bill Graham spoke about joint work with Sam Evens on the Belkale-Kumar cup product. Hugh Thomas spoke about combinatorial Littlewood-Richardson rules via jeu-de-taquin, a fundamental operation in the study of Young tableaux. Matthias Franz spoke about equivariant formality: what different mathematicians might mean by it (except rational homotopy theorists, who mean something completely different). After listing several related conditions that could be referred to as equivariant formality, he then described which are equivalent over field coefficients and which imply others. He finished by discussing what happens with integer coefficients. The morning was long, so the afternoon was short, mostly consisting of smaller conversations and some discussion of open problems.

Outcomes: As of this writing, several new connections and collaborations have been formed. Existing collaborations were energized; some started new work. At least one group of coauthors revised a paper in the light of presentations from mathematicians outside their field, which made them realize that their existing results were in fact stronger and more far-reaching than they had realized. One participant said it was the highlight of graduate school (so far); a very experienced senior faculty member said it was the best conference he had ever attended.

More concretely, conference attendees generated a list of open problems and an annotated list of references. The attendees also generated an annotated table of what's known and not known (to conference participants) in Schubert calculus. The references, open problems, and table are available on the conference wiki; we hope they will have a longer life as a larger public edits, corrects, and updates them.