Log minimal model program for moduli spaces

organized by
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Workshop Summary

This workshop was devoted to the birational geometry of moduli spaces, in particular the log minimal model program for moduli spaces of curves, surfaces, abelian varieties, and sheaves, as well as related problems in Geometric Invariant Theory (GIT). The first goal of the workshop was to synthesize very recent work of several groups of collaborators on the log minimal model program for moduli spaces, as well as related work on the GIT of Hilbert schemes and Chow varieties, and on the structure of the cone of effective divisors for punctual Hilbert schemes in the plane. The second goal of the workshop was to identify fundamental open problems in the field, and by bringing together different groups of people working in related fields, to find fresh perspectives on these problems. Towards this goal, we brought together researchers who work on the minimal model program, on moduli of curves and higher-dimensional varieties, and on analytic approaches to stability arising from the study of constant scalar curvature metrics.

Following the standard AIM model, two formal talks were given each morning, providing an overview of significant recent developments, emphasizing open problems and new techniques that might render these problems accessible. On Monday afternoon, Diane Maclagan moderated an open problem session which lasted the full afternoon. The problem session generated a list of questions, which the organizers arranged as a structured list divided into clusters focusing on various aspects of the theory. This formed the basis for the working groups that met in the afternoon on Tuesday, Wednesday, and Thursday. Finally, on Friday afternoon, working groups reported on the outcome of their investigations. The outcomes of the afternoon working group investigations are summarized below.

By general consensus, the workshop as a whole was both enjoyable and productive. The working groups made substantial partial progress on the focus questions of the workshop, and we expect several of the collaborations initiated here to continue in the coming months. We are grateful to AIM for providing the resources that made this workshop possible.

Summary of Working Group Presentations

(1) Study group on 'Nonexistence of asymptotic GIT compactification' by Xiaowei Wang and Chenyang Xu
(suggested by János Kollár, presented by Zsolt Patakfalvi)

This paper studies asymptotic Chow semistability of limits of families of canonically polarized stable varieties (in the sense of Kollár, Shepherd-Barron, Alexeev). Given a stable family of canonically polarized varieties over the punctured disc, this paper proves that if the stable limit is not asymptotically Chow semistable, then no limit is asymptotically Chow semistable. The group studied concrete examples of families of weighted pointed stable curves, using the fundamental inequality of Li and Wang to analyze the Chow semistable limit for successive multiples of the canonical polarization, and to explicitly see the non-stabilization of the Chow semistable limiting polarization. In these examples, the limiting curve is well-defined, and the limiting polarization approaches the canonical polarization as $r \to \infty$.

These examples raise the question of whether similar phenomena can be seen in the case of families of canonically polarized surfaces. For example, consider a canonically polarized family of surfaces degenerating to a surface with simple elliptic singularities (which is asymptotically unstable by results of Mumford). Is there a well-defined asymptotically Chow semistable limiting variety, with varying polarizing line bundle, or must we insert infinitely many bubbling components as the power of the line bundle increases? Can this limit be interpreted via K-stability?

(2) Birational models of moduli spaces of curves in small genus
(presented by Evgeny Mayanskiy)

The group studied Mukai's construction of genus six curves as a complete intersection

$$\mathrm{Gr}(2,5) \cap L \cap Q$$

where L is a codimension four linear subspace of $\mathbb{P}^9$ containing the Plücker embedding of the Grassmannian, and Q is a quadric hypersurface. They reported partial progress on describing the GIT quotient of codimension five quadrics intersected with $\mathrm{Gr}(2,5)$ under the automorphism group $\mathrm{SL}(5)$.

(3) Stability computations for Hilbert and syzygy points of curves
(presented by Anand Deopurkar)

Many researchers have sought to construct birational models of the moduli space of curves by doing Geometric Invariant Theory for Hilbert points. It would be interesting to adopt the same approach for syzygy points, which leads to the fundamental open problem of proving GIT-semistability of certain syzygy points. This group used computational techniques to prove the semistability of the syzygy point of a canonically embedded genus 7 ribbon with $\mathbb{G}_m$-action. This work is available at:
http://faculty.fordham.edu/dswinarski/genus7ribbon/

(4) Bridgeland stability conditions and birational models of the Hilbert scheme
(presented by Matthew Woolf)

This group sought to construct new moduli spaces not by varying the linearization on the Hilbert scheme, but by varying the birational model of the Hilbert scheme using Bridgeland stability conditions. First, the group focused on canonical curves of genus four, and tried to differentiate curves in the Petri divisor from the general genus four curve by using destabilizing complexes. Unfortunately, they did not succeed in identifying an appropriate complex.

Second, the group considered a lower-dimensional analogue of the same problem. They looked at stability properties of complete intersections of a quadric and a cubic in $\mathbb{P}^2$. The stability of intersections lying on a smooth or singular quadric could not be differentiated using line bundles on the Hilbert scheme, but it seems likely that they *can* be differentiated on alternate birational models as described in Arcara/Bertram/Coskun/Huizenga.

(5) Analysis of the nef and effective cones of the moduli space in $\mathrm{span}(\lambda, \delta, \delta_1)$ and $\mathrm{span}(\lambda, \delta, \delta_0)$.
(presented by Filippo Viviani and Natalie Durgin)

We have a simplicial nef cone:

$$\mathrm{Nef}(\overline{\mathcal{M}}_g) \cap \mathrm{span}(\lambda, \delta, \delta_0) = \langle \lambda, 12\lambda - \delta_0, 10\lambda - 2\delta + \delta_0 \rangle \,.$$

This group studied the question of whether the divisors on the edge of this cone are semiample. It is known that λ gives the morphism to the Satake compactification of $\mathcal{A}_g$, that the edge spanned by λ and $12\lambda - \delta_0$ gives a morphism to the perfect/second Voronoi compactification of $\mathcal{A}_g$ (by work of Gibney), and that the edge spanned by $12\lambda - \delta_0$ and $10\lambda - 2\delta + \delta_0$ gives the morphism to the moduli space of pseudostable curves.

The group raised the question of whether $12\lambda - \delta_0$ is semiample for all g (this is known to be true for $g = 3$ by work of Rulla), and whether it is possible to interpret the remaining edge of the two-simplex geometrically.