Macaulay2: expanded functionality and improved efficiency

organized by
David Eisenbud, Claudia Polini, Claudiu Raicu, and Emily E. Witt

Workshop Summary

The theory of Gröbner bases has proved over the years to be an effective tool for the study of systems of polynomial equations. The computer algebra system Macaulay2, developed by Grayson and Stillman since the early 90s, is built around the theory of Gröbner bases, and has been widely used by researchers in commutative algebra and algebraic geometry. The Macaulay2 website lists more than 2800 articles referring to the Macaulay2 system, but this likely illustrates only a fraction of the impact the software has had over the past 30 years. There is nowadays a thriving Macaulay2 community, with many early-career researchers joining forces every year with more established ones during Macaulay2 workshops, where they develop new packages and expand the capability of the software. The American Institute of Mathematics provided an ideal venue for the workshop *Macaulay2: expanded functionality and improved efficiency*, where several important directions have been initiated.

The workshop featured short presentations during the morning sessions, while the bulk of the rest of our time was spent implementing code and documenting packages. Early in the week the presentations focused on explaining the background and the main goals of the projects, while later on they described the progress being made and the challenges that each group encountered. Intensive discussions took place throughout the week, both in terms of clarifying the mathematical issues, but also regarding the translation from mathematics to code.

The environment of the American Institute of Mathematics was perfectly aligned with the activities and resulted in a very successful Macaulay2 meeting. In what follows we summarize some of the goals and progress made by several working groups participating in the workshop. An accompanying file includes an extensive list of open problems and projects that was compiled with input from the participants during one of the sessions.

Working group on Sheaf Cohomology..

Morphisms of sheaves are fundamental mathematical objects that had not been implemented in Macaulay2. The group had the goal of implementing the type "SheafMap", which led to some subtle discussion on how to choose the "best" representative of a morphism of sheaves along with the most efficient way to store the data of a morphism of sheaves. One of the key features of a morphism of sheaves is that there is an induced map on cohomology, which can be very difficult to compute concretely by hand. For this reason, the main goal during the AIM workshop was to have morphisms of sheaves along with their induced maps on cohomology implemented in Macaulay2.

In the end, the group was successful in implementing morphisms of sheaves in Macaulay2 with essentially all of the functionality that morphisms of R-modules have, including the "prune" function and essentials such are kernels/images/cokernels. The group also updated many functions defined on the type "Sheaf" to now behave functorially and was successful at implementing induced maps on cohomology (more generally, induced maps on global Ext). Future goals for this project include a complete overhaul of the pre-existing code for projective varieties along with multigraded implementations of sheaf maps for normal toric varieties. An ultimate goal is to have complexes of sheaves implemented with all of the same functionality that complexes of R-modules currently enjoy in Macaulay2.

Working group on DG Algebras..

The DGAlgebras package provides functionality for performing computations with differential graded homological algebra. Differential graded techniques allow one to study finiteness properties of free resolutions of modules over non-regular rings, many of which are a priori infinite objects.

The main goal of the group was twofold: to make the existing functionality a bit easier to use, and to add features related to differential graded modules. In particular, the group worked on adding semifree resolutions of modules and complexes of finitely generated modules over a DG algebra. This will allow researchers to investigate additional finiteness properties of infinite free resolutions, especially those related to Golod and complete intersection rings.

Working group on Parallelization..

The parallelization group at AIM worked on understanding issues with the current implementation of the front-end parallel programming with threads and tasks, internal parallelization bottlenecks, and the Macaulay2 core alterations and package using Message Passing Interface. The latter would potentially enable Macaulay2 execution on large heterogeneous distributed clusters.

Progress was made on all of the above during the workshop, some of the effort continued on Doug Torrance's parallelization branch on his fork of Macaulay2 on github. In particular, parallelApply command was added by Dave Barton and shows good speedups on examples.

Working group on Direct Summands..

The AIM group on computing direct summands focused on examining the summands of syzygies of the residue fields of Artinian rings. Previous work of Dao—Eisenbud had shown that one often gets the residue field as a summand of some not-so-high syzygy. Focusing specifically on Artinian rings with maximal ideals generated by two elements, the group observed some surprisingly simple behavior of these summands, which was essentially that in the decomposition of all syzygies, there are copies of the residue field, the maximal ideal, and one other module appearing as a direct summand of the second syzygy module. After the AIM workshop, Devlin Mallory and Mahrud Sayrafi have implemented an algorithm to find direct sum decompositions, and written a paper documenting their results, which has been accepted for the upcoming MEGA conference (Effective Methods in Algebraic Geometry) in Leipzig, Germany.

Because of the above observations, at AIM Dao, Eisenbud and Polini conjectured that all possible syzygies of the residue field of an Artinian ring of embedded dimension 2 were indeed direct sums of the residue field itself, the maximal ideal, and just one more indecomposable module. This surprising conjecture was proven during the special semester at SLMath. More precisely, one can write every syzygy of the residue field just using the residue field, the maximal ideal, and the dual of the maximal ideal. The resulting paper will acknowledge the support of AIM. The conjecture would not have been possible without the direct summand package by Mallory and Sayrafi.

Working group on local cohomology, b-functions, $\mathcal{D}$.
-modules.

The objective of the local cohomology working group was to improve the usability and performance of the local cohomology routines inside the D-modules package of Macaulay2. Some participants in the workshop raised the issue that the options for the different routines were not properly documented, and that computations of very simple examples usually took much longer than expected.

The group identified the fact that in most local cohomogy computations, the whole Čech complex was being computed, even when the user was only interested in only one or two cohomological degrees. This issue was fixed, and a pull-request was created to merge the changes into Macaulay2's official library. The documentation of the different methods was also updated for improved usability.

The other bottleneck in the implementation of the computation of local cohomology modules is the computation of the annihilator of the $\mathcal{D}$-module associated with f^s, for f a ring element. Further investigation found that the algorithms implemented in Macaulay2 for this computation are not the most efficient. However, the kernel of Macaulay2 currently lacks the features to implement the more optimized algorithms, as they use so-called Ore algebras. Implementing Ore algebras in the kernel of Macaulay2 was out of the scope of the workshop due to time constraints, but it is a pending project that would greatly improve the performance of $\mathcal{D}$-module computations, and local cohomology modules in particular.