Mahler's conjecture and duality in convex geometry

organized by
Alexander Koldobsky, Dmitry Ryabogin, Vladyslav Yaskin, and Artem Zvavitch

Workshop Summary

The goal of the workshop was to consider several problems related to the duality of convex bodies. The main directions included the Mahler conjecture on the volume product of convex bodies, the duality between sections and projections in convex geometry, and the duality of covering numbers. As the result of morning lectures and afternoon discussions, the participants were able to identify many open problems and promising ways to attack them. The workshop has created new collaborations which will undoubtedly lead to new results in the area.

The volume product of a convex body K in R^n is defined by $P(K) = Vol_n(K)Vol_n(K^*)$, where K^* is the polar body of K. Mahler's conjecture asks whether the minimum of the volume product in the class of origin-symmetric convex bodies is attained at the unit cube. Despite many important partial results, the problem is still open in dimensions 3 and higher. The celebrated result of Bourgain and Milman establishes Mahler's conjecture up to an absolute constant. At the workshop, several participants reviewed recent results related to the conjecture. Meyer gave a comprehensive survey of the history, known results and approaches. Kuperberg explained his recent differential geometric proof of the Bourgain-Milman theorem giving an explicit (not optimal) constant. This approach was extensively discussed by a group of participants during afternoon sessions. Reisner described several results for establishing the conjecture for certain classes of bodies. Barthe considered the case where bodies have many symmetries. The latter case was discussed in the afternoon sessions in the setting of Coxeter groups and a lecture on these groups was given at the request of the participants by Zobin. König suggested a version of the Mahler conjecture on the sphere which drew instant interest from the participants and initiated several discussions.

The concept of $\mathrm{as}_p(K)$, the p-affine surface area of a convex body K, was discussed in the talks of Werner and Stancu. They presented various inequalities for such areas. The importance and relevance of these inequalities to the topic of the workshop comes from the fact that $\mathrm{as}_p(K)$ gives the volume of K^* as $p \to \pm\infty$. In particular, inequalities for p-affine surface areas allow to obtain estimates for the volume product of convex bodies.

It has been known for a long time that many results on sections and projections of convex bodies are dual to each other, in the sense that sections of a body behave in a similar way to projections of the polar body. Gardner summarized this information in his lecture by introducing a dictionary for transforming results about sections into results about projections. Much of the discussion was centered around this dictionary. In particular, the participants were trying to find a result for projections whose translation into the language of sections would not hold. Despite the efforts of a group of participants, such results were not identified, which added more to the mystery of duality between sections and projections. However, the discussion led to several new questions, which if solved will bring more light to this direction. In particular, the participants tried to give a new proof to the result that every polar projection body is an intersection body, which would be based on a functional analytic characterization of these classes of bodies given by a result of Goodey, Lutwak and Weil.

Gardner reminded the participants of an old question, going back to Bonnesen's 1926 paper, that asks whether a convex body is uniquely determined by its inner section function. For origin-symmetric bodies the answer to the problem is known to be affirmative, but without the symmetry assumption the problem is still open in dimensions $n \geq 3$. Gardner shared his thoughts on this problem and, in particular, on possible constructions that might lead to a counterexample. Additional interest to this problem was inspired by the new definition of intersection bodies, involving duality, presented by Meyer in his lecture.

An overview of the duality conjecture for covering numbers was given by Tomczak-Jaegermann in her lecture. For two centrally-symmetric convex bodies K and L in $\mathbb{R}^n$, the covering number $N(K, L)$ is defined as the smallest number of translates of L needed to cover K. An old problem of Pietsch asks whether $N(K, L) \leq N^a(L^*, bK^*)$ for some absolute (independent of the dimension and of the bodies K, L) constants a, b (the so-called "duality conjecture"). It was observed by König and Milman that $N(K, L) \leq C^n N(L^*, K^*)$, where C is an absolute constant. Recently Artstein, Milman and Szarek have provided the affirmative answer to the conjecture for the case when one body is an ellipsoid. This result has been extended by the same authors and Tomczak-Jaegermann to the case when one body is K-convex (equivalently, the corresponding normed space has a non-trivial type). Later, combining two known results, E. Milman has noticed that the duality conjecture is true up to logarithm, namely it holds with $a = C \ln n \ \ln\ln n$ and $b = C/\ln n$, where C is an absolute constant. During the discussion session the participants have tried to construct a counter-example to the conjecture using the product of the Walsh matrix and a diagonal one.