Integro-differential equations in many-particle interacting systems

organized by
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Workshop Summary

Overview

The workshop focused on the mathematical theory of nonlinear partial differential equations that model various physical phenomena in gas dynamics, plasma physics, quantum statistics, and biology. These models pose significant challenges, including nonlinear behavior, nonlocal effects, the dynamics of key quantities, and the formation of singularities in finite time. In recent years, this research area has seen substantial progress. The workshop aimed to bring together active researchers from these intersecting fields to identify new problems and discuss promising long-term research directions. The main focus of the workshop was on four interrelated research topics: (1) Conditional regularity and singularity formation for inhomogeneous kinetic equations, (2) Hele-Shaw problem, (3) Free boundary problems, (4) Wave kinetic equations.

We had participants from universities in the United States and Canada at all career stages. Several participants were well known experts in the community of kinetic theory and nonlocal partial differential equations. A few participants were also working on dispersive equations with application in surface waves and quantum mechanics. A few junior participants were working on these problems from the numerical prospective.

Problems discussed in work groups

During the open discussion on the first day of the workshop, a list of twenty six questions was proposed by the participants. Later in the day, the organizers narrowed down the list of questions to eleven, making sure that topics included a range of interests present among the workshop participants. Participants later chose a group to join. In total, six groups were formed, and for the most part the groups remained the same throughout the workshop.

- One group was focused on the study of the well-posedness of the relativistic Landau equation, inspired by the recent breakthrough results on the monotonicity of the Fisher information for the classical Landau equation. After reviewing literature, both for the relativistic equation and its classical counterpart, the group first discussed a possibility of proving the monotonicity of the Fisher information for a simpler relativistic model – the relativistic linear Fokker-Planck equation in dimension one. Once those calculations looked promising, the group turned focus to the relativistic Landau equation itself. The team has continued working on the project, and is hopeful that the tools developed in this project may be used for the study of other relativistic kinetic equations such as the relativistic Boltzmann equation.
- One group was discussing the asymptotic behavior of the Fokker-Plank equation, with potential applications to sampling acceleration, which would require nonlinear modifications of the equation. Their aim is to analyze a new sampling algorithm

that incorporates a preconditioner matrix, which plays the role of a covariance matrix in the nonlinear Fokker–Planck equation. The central question they are studying is whether they can construct a suitable metric such that the solution to the nonlinear Fokker–Planck equation can be interpreted as the Wasserstein gradient flow with respect to this metric. The group has identified two promising directions worth pursuing: (1) the Jordan–Kinderlehrer–Otto scheme; (2) the framework of Evolution Variational Inequalities or Energy Dissipation Inequalities.

- One group discussed global well-posedness for the inhomogeneous Boltzmann equation. They are interested in generalizing a recent result by one of the workshop participants (Dominic Wynter), where large data global-well posedness for the nearly homogeneous cutoff Boltzmann equation in spatial dimension one and velocity dimension three was established for the cutoff case that removes singularity of the collision kernel. The group interested in extending this result to the non-cutoff case. The group also discussed potentially extending results to a model from wave turbulence, namely, the wave kinetic equation.
- One group was focused on the study of the one-dimensional nonlocal transport equation. This is an evolution equation for a scalar whose partial derivative with respect to time is given by a linear integro-differential operator. The linear integro-differential operator is conceived of as a nonlocal analogue of the derivative of a scalar along of a vector field. A first problem raised was determining the behavior of such linear equations in the singular limit where one is expected to recover the standard transport equation. The group also thought about how the nature of the equation breaks the usual Lagrangian formulation picture in the local setting, giving up the usual notion of characteristics and even of finite speed of propagation.
- One group discussed the issue of the implosion for kinetic equations. During the first day the participants brainstormed about existing literature and the possibility of constructing kinetic toy models that exhibit blow-up in finite time. In the subsequent days the participants were able to tackle a series of problems for which the question of formation of singularity in finite times make sense. The participants also discussed the implication of these results for related equations in continuum mechanics. The group parted with a list of bullet points that they intend to discuss in the near future.
- One group was exploring the Hele-Shaw equation, which is a free-boundary problem that is known to admit a formulation as a nonlinear integro-differential parabolic equation. One question the group first considered is whether there is a particle system that converges to this equation. This has potential applications in modeling crowd motion. The group also considered the related question of studying the parallels between the nonlocal operator behind the one-phase Hele-Shaw equation and degenerate nonlinear local equations like the p-Laplacian and Eikonal equations. One expectation is that these parallels might provide insights into questions such as existence and regularity.

Summary of Talks

- On Monday we had two overview talks on fundamental mathematical models in plasma physics and quantum mechanics: Wave equations and kinetic equations of Landau and Boltzmann type. Ioakeim Ampatzoglou started the day by introducing the kinetic wave equation and presented several results on well-posedness and

scattering. Stanley Snelson presented inhomogeneous kinetic equations and the key mathematical techniques used in their analysis. He demonstrated how classical methods—such as the maximum principle and compactness arguments—can be adapted to this non-standard class of problems. He also outlined the major open problems in the field, highlighting those that remain the most challenging.

- Tuesday was dedicated to presentations on free boundary problems and integro-differential equations. Inwon Kim spoke about equations modeling interface separation between regions of high and low aggregation density—a phenomenon known as phase separation. These models involve systems of partial differential equations for the density function, which, in the limit, give rise to a free boundary approximation that captures the evolution of the interface. Russell Schwab's talk focused on a non-local Jacobian equation, motivated by the challenge of developing Aleksandrov-type estimates adapted to integro-differential equations. One discussion group explored how such estimates might be extended to kinetic equations, where transport effects play a significant role.
- The talks on Wednesday explored the relation between microscopic and macroscopic models. Jiajie Chen explained how to construct blow up (implosion) solutions for compressible Euler and for other related models. The talk served as a review for some groups who were starting a project on construction of blow-up solutions for kinetic Landau equation. Jacob Bedrossian instead, explained how solutions to kinetic equation can have mild shock profiles. Intuitively, one could think that a shock visible at the macroscopic level can be observed, in some sense, also at the kinetic level.
- On Thursday Weiran Sun introduced the M-operator approach and explained how this operator helps in the study of uniqueness of continuous or bounded solutions for a broad class of Landau-type nonlinear kinetic equations. Weiran explained how this novel method enables to work with solutions of low regularity. After the talk several participants gathered at the board and asked questions related to possible continuation and extension of the method to dispersive equations. Franca Hoffmann completely changed prospective and explained the role of partial differential equations in optimization and sampling. She also explained the most important methods used in uncertainty quantification and in parameter optimization.
- On Friday morning, Yukun Yue gave a talk on damping instabilities in solutions to the Vlasov–Poisson system, a fundamental model in plasma physics. He discussed how instabilities in the system can be mitigated through the influence of electric fields. This topic is of significant interest in the community due to its direct applications in particle accelerator technology and nuclear fusion research.

Concluding Remarks

The workshop was very well received, with enthusiastic feedback from junior participants who highlighted how much they learned over the five days. Several new collaborations were initiated during the event, two of which are already at an advanced stage.