Markov chain mixing times

organized by
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Workshop Summary

1 Introduction

In 2004 the American Institute of Mathematics hosted a workshop on *Sharp thresholds for mixing times*. At that time the quantitative analysis of Markov chains was already a well established research topic, with important connections to Markov Chain Monte Carlo (MCMC) algorithms and to other areas of Mathematics such as Analysis, Geometry, Representation Theory and rigorous Statistical Mechanics.

The twelve years since that workshop have seen amazing progress on mixing times. A substantial number of these advances trace their origins to AIM activities. At the same time, many of these recent results rely on ideas that were not foreseen then. Somewhat surprisingly, new ideas have often come from within Probability Theory itself. Techniques from *Interacting Particle Systems, Maximal Inequalities and Coalescence and Fragmentation* have been used in novel and subtle ways, yielding proofs of some old conjectures and raising new questions such as the dependence on the initial condition.

2 Aims and scope

Our main goal was to bring together experts on these topics in order to deepen our understanding of these connections and to find new paths for further progress in the subject. In particular there were a number of talks to discuss some of the new techniques mentioned above which have emerged in recent years in connection with mixing times. Our second goal of this workshop was to put forward a research agenda for the future of the area, which could be used as a roadmap for how the theory is going to be developed in the near future. In particular, some of the main problems which motivated us to organise this workshop were on: (a) characterization of cutoff (give necessary and sufficient conditions for the cutoff phenomenon to take place for reasonably general families of Markov chains), and (b) robustness of cutoff with respect to perturbations of the underlying Markov chain, and (c) how does cutoff depend on the choice of the initial condition?

3 Summary of Talks

Starting with the first of these two goals, on the first day, Lubetzky gave an overview of his work with Sly in Information Percolation and how this can be used to prove sharp results for spin systems, while also outlining some intriguing open questions.

Peres gave an introduction to maximal inequalities, and how these arise naturally in connection with the mixing time. Later on, Hermon explained how these ideas were applied to resolve a conjecture of Aldous and Fill on the equivalence betweeen cutoff for a continuous

chain and a chain averaged over two steps. Sousi described her work with Boczkowski and Peres on Eulerian graphs, which are oriented graphs such that the in-degree equals the out-degree. In such graphs the conventional wisdom is that many properties are similar to reversible Markov chains. They were able to obtain a general bound on the mixing time on such graphs in terms of the size of the graph. Strikingly, they show an example where small perturbations in the laziness of the chain can have drastic effects on the mixing time of the walk.

Lacoin gave a high-level overview of his proof of cutoff for one-dimensional exclusion and stirring processes. Olivieira discussed aspects of his study of the mixing times of exclusion and interchange on general graphs, drawing on Morris' chameleon process.

Pillai discussed his work with Smith on Kac's random walk on the n-sphere and related walk on $SO(n)$. Berestycki described the idea of his proof of cutoff with Sengul for random walks invariant under conjugacy on S_n. These two talks, scheduled back to back, revealed similarities between the two approaches that demand to be explored in greater depth.

Cannon discussed her work with Randall on Glauber dynamics for random tilings, mentioning several open problems which were discussed in groups. Ben Hamou discussed the question of whether nonbacktracking random walk mixes faster than simple random walk, which also led to some of the open problems being discussed.

Finally, Diaconis gave an inspirational lecture on some of his favorite problems in the field, including many concrete examples. This lecture led to a lively discussion on the nature of the cutoff phenomenon, Diaconis arguing that it should be related to the fact cutoff takes place when there are no nontrivial low-dimensional representations. Peres observed that another reason why this might happen (particularly when there is no nice algebraic structure) is when the mixing of the chain depends on many random variables that are nearly-independent, so that cutoff is then related to concentration of measure in infinite dimensions.

Taken together, these excellent lectures gave a lucid exposition of some of the most recent techniques being applied to the theory of mixing times, as well as a highly stimulating picture of the next challenges.

4 Summary of working groups

Regarding the second goal, on the first day Amir Dembo and Perla Sousi moderated two extensive problem sessions. This helped the organisers select around ten problems to be studied in groups. Around six problems were selected.

(1) Long-range correlation of noisy majority with small noise? Mixing exponentially slow?
(2) Reversing slows down mixing, nonbactracking random walk mixes faster?
(3) Biased transpositions and sensitivity of cutoff for bounded perturbations (especially on groups).
(4) Diagnostics *nlogn* universal lower bound for mixing of Potts model.
(5) Tilings.
(6) Cutoff on the small world.
(7) Kac's walk.

Several ideas emerged from these discussions, including partial progress in several groups. We give several examples below. The group on the effect of making a chain reversible managed to construct a counterexample to the initial conjecture (which was that mixing must always occur faster for the nonreversible process). Another counterexample was constructed to show that nonbacktracking random walk may take longer to mix than the standard kind.

The group on biased transpositions discussed a general argument showing that mixing is not delayed by more than a constant factor when making bounded perturbations to the generator of a group. In the case of random transpositions with a weight on each transposition, they seem to have proved cutoff.

The diagnostics group manage to crystallize the essence of its questions into two problems. The first one is this: for attractive systems, such as the Ising model, is it true that the full coalescence time in the natural monotone coupling is always at most twice the mixing time? And second: for the Potts model, can one distinguish $nlogn$ from e^{cn} mixing times just by observing the dynamics?

Finally, the Kac's walk group managed to relate their basic problem – namely, optimal bounds for the $SO(n)$ random walk – in terms of a random matrix problem. It is believed that the current best bounds for the mixing time of this random walk will be improved with a more careful analysis of this walk.

5 Concluding remarks

Based on the initial feedback from participants (especially the younger researchers) and the friendly working atmosphere during the week, we believe the workshop was a true success. It is expected that a number of new collaborations may result from the workshop and at least a few papers should grow out of the problems considered.