Metric embeddings

organized by
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Workshop Summary

The *Metric Embeddings* workshop, held at the American Institute of Mathematics in July 2025 was the first of its kind in almost 15 years. As such, it became a forum of very lively discussions among the participants. We hope that interactions initiated during the workshop will lead to many developments in the field, both short term and long term. Below is a summary of the scientific content of the workshop.

Talks. Each day of the workshop started with 2 lectures of almost 90 minutes each, with the exception of Friday when we had 3 lectures. These featured some of the most exciting recent progress in all aspects of embedding theory with topics including nonlinear spectral gaps, vector-valued concentration, embeddings of transporation cost spaces, embeddings of uniform Roe algebras, embeddings without shrinking, Lipschitz light and bounded length distortion embeddings, as well as connections to topics in theoretical computer science (nearest neighbor search, evaluating average distance in metric spaces, the sparsest cut problem, and the k-server conjecture) and geometric group theory. Each talk was characterized by an informal style, including lively discussions between the speaker(s) and the audience.

Open Problem Sessions. A key goal of this workshop was to revive the influential list of "Open Problems on Embeddings of Finite Metric Spaces"

(https://kam.mff.cuni.cz/∼matousek/metrop.ps), which was maintained until 2011 by Jiri Matoušek, by recording progress on existing problems, and adding to it a list of new questions and challenges. This is sure to be a catalyst for future developments in the field. To curate the updated list, we organized a series of (plenary and specialized) open problem sessions during the week as well as some more targeted breakout working groups to discuss at length plausible approaches to open problems that arose as central in the open problem sessions. These sessions were organized as follows:

Monday and Thursday plenary open problem sessions. These were the main discussion sessions of the workshop, where all participants gathered at the lecture room and proposed open questions. The Monday session (moderated by Yuval Rabani) led to a selective list of 31 open questions covering all aspects of embedding theory. Along the way, the discussion also revealed progress on questions from Matoušek's original problem list. On the Thursday plenary session, participants got together again to record preliminary thoughts on these 31 problems as well as some related subproblems that were discussed in detail in the specialized open problem sessions of Tuesday. This final problem list, curated by Christopher Gartland, is different from the forthcoming update to the Matoušek list and is currently featured on the AIM website at http://aimpl.org/metricembeddings/. A non-exhaustive selection of topics touched upon in this problem list includes: embeddings of unions of spaces, expanders

with respect to metric spaces, coarse and linear universality, Markov type, embeddings of groups and transportation cost spaces, online embeddings, embeddings into distributions of dominating tress, computing spread constants, and many others.

Tuesday specialized open problem sessions. On Tuesday, the participants were split in 3 more targeted groups to discuss more specialized open problems and potential approaches towards them. The 3 groups were focused on problems in Analysis (moderated by Keith Ball), Geometry (moderated by Robert Young) and Theoretical Computer Science (moderated by Alexandr Andoni). These led to more technical discussions that contributed to the final form of the problems featuring in the open problem list.

For example, Thomas Hutchcroft suggested the question of whether there is an interesting coarse analogue of Property (T). Follow-up questions were suggested by Assaf Naor, who suggested the candidate property that random graphs form metric expanders, and asked whether every finitely generated group admit a sequence of metric expanders for it, and Guoliang Yu, who asked whether a group with no property (T) have expanders with respect to it, and also asked whether there is a finitely generated group such that every finite metric space coarsely embeds into it, with uniform moduli.

Wednesday and Friday breakout groups. On Wednesday, the participants were split in 3 groups to discuss possible approaches to tackle two problems which arose as central in the open problem sessions.

• On the Monday open problem session, Keith Ball proposed the long-standing open problem of *whether a normed space with non-trivial Markov type is superreflexive*. This question has been resisting attempts since its formulation in his 1992 classical paper on extensions of Lipschitz functions. A group of around 15 experts discussed this for almost 3 hours and various original approaches (to both prove it and disprove it) were proposed. These include trying to come up with a variant of Pisier's classical renorming method using stationary Markov chains instead of martingales and coming up with a sequence of test graphs for superreflexivity which have bounded degrees but unbounded girth. As a probably tractable and surely insightful first step, it would be worthwhile to compute the Markov type of the James nonsuperreflexive space of Rademacher type 2.

• Following the Monday morning lecture of Christopher Gartland and Mikhail Ostrovskii who showed that the L_1 distortion of the transportation cost space over $\{1,\ldots,n\}^2$ is of the order of $\log n$, the audience immediately raised the question of *evaluating the distortion of the transportation cost space over the higher dimensional grid* $\{1,\ldots,n\}^d$, where $n,d \geq 2$. It is natural to conjecture that this is of the order of $d\log n$, as this would also serve as a generalization of a result of Khot and Naor. As a possible way of attacking this, it was proposed that a higher-dimensional version of the L^1 Poincaré inequality proven by Gartland and Ostrovskii may be true, and to prove this, a corona decomposition adapted to the L^1 Poincaré inequality was proposed. This has resulted in the preprint entitled "L_1 -distortion of Earth Mover Distances and Transportation Cost Spaces on High Dimensional Grids" (https://arxiv.org/abs/2602.19434) by Chris Gartland, Mikhail Ostrovskii, Yuval Rabani, Robert Young, all participants of the workshop.

• Finally, a third breakout room moderated by the organizers was created in order to *answer questions* (mainly by graduate students and postdocs) on open problems which were

mentioned on Monday and Tuesday. This facilitated their introduction to the discussions held in the other two rooms.

On Friday, the participants were split in many smaller unmoderated discussion groups. The Wednesday discussions on Markov type and transportation costs were continued by subsets of the original groups. A reasonably large group was spontaneously formed to discuss *variants of the Lang–Plaut problem*, which asks whether every doubling subset of Hilbert space embeds bi-Lipschitzly into some finite-dimensional Euclidean space. One such variant asked was whether a sharp version of the Assouad embedding theorem with optimal distortion and dimension held, and various ways to attack this for doubling subsets of Hilbert space were proposed.

Future Plans. We are optimistic that the Metric Embeddings workshop will be a catalyst for future developments in the field. It clearly showed the necessity of holding regular meetings in the area as even a week with many discussion sessions was not nearly enough to touch open all the topics which we would have wanted. Our first goal is to create a major revision of Matoušek's open problem list to reflect the developments and new challenges that were put forth during the workshop. At a later time, we will start planning on how to turn this into a regular event, thus allowing the community to meet every few years.

Bibliography