The minimal model program in characteristic p

organized by
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Workshop Summary

The focus of the workshop was the new progress in birational geometry in characteristic p which has received a lot of impetus from recent advances in commutative algebra. The workshop brought together experts in both areas. As usual for an AIM workshop, there were two talks in the morning, with an hour long gap in between for discussion. Typically one talk each morning was by someone with a background in commutative algebra and the other by someone in birational geometry.

The key new feature in characteristic p is the failure of Kodaira vanishing, or more generally Kawamata-Viehweg vanishing. Many of the talks focused on the Grothendieck trace map of some power of the Frobenius morphism F

$$Tr\colon\ mapF^e_*\omega_X.\omega_X..$$

For example, this map induces a map on global sections

$$mapH^0(X, ringX.(mp^e(K_X+\Delta)).H^0(X, ringX.(m(K_X+\Delta)).$$

and one defines $S^0(X, ringX.(m(K_X+\Delta))$ to be the image of this map for any e sufficiently large. If one twists by an ample divisor, this canonically defined subvector space enjoys many of the properties of the space of all pluricanonical forms in characteristic zero; one can often lift these sections as an application of Serre vanishing. The remaining talks were devoted to interesting features of birational geometry in characteristic p. One highlight was Hacon's discussion of his recent work with Xu on the existence of flips for terminal $\mathbb{Q}$-factorial projective threefolds and the construction of minimal models for smooth projective varieties such that K_X is pseudo-effective.

On the first afternoon the workshop generated many interesting open problems, centred around seventeen different themes. On the remaining four afternoons, the participants formed four groups and attacked a suitable selection of these problems.

Lifting regular $n-1$ forms to a resolution:

Let (X,Δ) be a log pair and let $\pi\colon\ mapY.X.$ be a log resolution. It is a general result of [GKK10] that in characteristic zero one can lift regular $n-1$ forms to regular $n-1$ forms on Y, provided that $K_X+\Delta$ is kawamata log terminal, and to $n-1$ forms with a log pole on Y, if $K_X+\Delta$ is log canonical. This group explored to what extent this is true in characteristic p.

If X is the cone over a Fano variety D, embedded by some multiple of $-K_D$, then X is $\mathbb{Q}$-factorial kawamata log terminal. Consider the exact sequence,

$$ses\Omega^{[n-1]}_Y.\Omega^{[n-1]}_Y(\log D).\Omega^{[n-1]}_D.,$$

where square brackets denotes the double dual. Taking global sections, note that if

$$Ker\ mapH^0(D, \Omega_D^{[n-1]}).H^1(Y, \Omega_Y^{[n-1]}).,$$

is non-zero then the map

$$mapH^0(Y, \Omega_Y^{[n-1]}).H^0(Y, \Omega_Y^{[n-1]}(\log D)).,$$

is not surjective, which is to say that there are global $n-1$ forms on X which lift to $n-1$ forms on Y with log poles along D.

Now, using a construction of Kollár, one can find a Fano variety D in characteristic p, which is mildly singular, such that $H^0(D, \Omega_D^{[n-1]})$ is non-zero (in fact $\Omega_D^{[n-1]}$ even contains a big line bundle). On the other hand, it is not hard to see that

$$mapH^0(D, \Omega_D^{[n-1]}).H^1(Y, \Omega_Y^{[n-1]}).,$$

is zero if D is embedded by the pth power of some line bundle.

There are similar but more complicated examples to show that if X is log canonical then not every form with log poles lifts to a form with log poles. In this case D is a Calabi-Yau variety.

It is also natural to consider what happens if (X, Δ) is globally F-regular, since this is more restrictive than saying that (X, Δ) is kawamata log terminal; unfortunately there wasn't enough time to consider this interesting question.

Rational connectedness of globally regular threefolds:

In characteristic zero it is known that every Fano variety is rationally connected. In characteristic p it is natural to ask if every F-regular variety is rationally chain connected, that is, any two points can be connected by a chain of rational curves.

This group looked at the case when X is a terminal $\mathbb{Q}$-factorial threefold in characteristic $p > 5$. Using the recent work of Hacon and Xu, [HX13], one can run the K_X-MMP and by work of Cascini, Tanaka and Xu, we know that this will terminate with a Mori fibre space. Let $f\colon\ rmapX.Y.$ be one step of the K_X-MMP. If f is birational one can show that if Y is rationally chain connected then so is X, and conversely if X is F-regular then so is Y.

So we might as well assume that f is a Mori fibre space and the problem breaks into cases based on the dimension of Y. If Y is a point then there is a quite general argument to show that Y is rationally chain connected. If Y is a surface then it must be a rational surface, which is rationally connected. Restricting to a general rational curve in Y, we have a ruled surface S over a rational curve, which is rational by Tsen's theorem, so that S is rationally connected. But then X is rationally chain connected.

The only remaining case is when Y is a curve. The fibres are rational surfaces. If the fibres are not $\ pr2.$ then they are ruled and after base change we reduce to the case above. When the fibres are $\ pr2.$ we can apply a theorem of de Jong and Starr, which is a generalisation of Tsen's theorem to the case of higher dimensional fibres which are separably rationally connected.

Grauert-Riemenschneider in characteristic p:

By examples of Raynaud [Raynaud78], and later Lauritzen and Rao [LR97], it is known that Kodaira vanishing fails in positive characteristic. Deligne and Illusie in [DI87] show that if one can lift to $W_2(k)$ then Kodaira vanishing holds; in particular they gave the first algebraic proof of Kodaira vanishing in characteristic zero.

The failure of Kodaira vanishing, and more generally Kawamata-Viehweg vanishing, was considered a major stumbling block to progress in the minimal model program in positive characteristic. Recent advances were made possible by noticing that asymptotic Serre vanishing together with the Frobenius morphism is often an adequate replacement of Kodaira vanishing.

The goal of this group was to investigate the status of a related vanishing theorem, namely Grauert-Riemenschneider vanishing in positive characteristic. This asserts that for any proper birational morphism $\pi\colon\ mapY.X.$ of varieties with X smooth, the higher derived images $R^i\pi_*\omega_Y$ vanish for $i > 0$, where ω_Y is the dualising sheaf on Y. It is well known, and explicitly carried out by Kovács and Hacon in [HK12], that by taking an affine cone X over a counteraxample to Kodaira vanishing yields a counterexample to Grauert-Riemenschneider vanishing for the blow up of the vertex of the cone. However, X is highly singular (in particular not CM) in this case and so it is natural to ask if Grauert-Riemenschneider holds if one restricts the singularities of X. In a highly nontrivial paper Chatzistamatiou and Rülling [CR11] recently showed this if X is smooth. A natural next class of singularities to consider would be F-regular varieties, but during our discussion no obvious line of attack to this question crystallised.

However one may ask if there is an asymptotic version of Grauert-Riemenschneider vanishing: Concretely, the canonical sheaf ω_Y comes equipped with the trace of the Frobenius $Tr\colon\ mapF_*\omega_Y.\omega_Y..$ Pushing forward to X this induces, for all $i \geq 0$ maps

$$mapF_*R^i\pi_*\omega_Y.R^i\pi_*\omega_Y.$$

and one may speculate that a high enough iterate of this map is zero for $i > 0$. Indeed, revisiting the examples of a cone X over any smooth projective variety V and taking π to be the blowup of the vertex, a simple calculation shows that this is indeed the case, and is in fact a consequence of asymptotic Serre Vanishing on V.

This asymptotic Grauert-Riemenschneider vanishing is intimately related to an asymptotic version of the Kawamata-Viehweg vanishing theorem which was considered before by several authors. Explicitly, Langer asks in [Langer11], drawing on previous work by Bhatt [Bhatt12], if for a normal projective variety X and a nef and big line bundle L the higher cohomology $H^i(X, L^{-n})$ vanishes for $i > 0$ and $n \gg 0$. Langer shows that this is true for H^1.

Lower bounds for the Seshadri constant:

This group investigated possible approaches to finding lower bounds, only depending on the dimension, for Seshadri constants (at general points) in positive characteristic. By the results in [MS12], such bounds would imply that for every ample line bundle on a smooth projective variety X, the line bundle $\omega_X \otimes L^m$ gives a birational map for all integers m greater than a bound only depending on $\dim(X)$.

The group focussed on the case of surfaces and started by discussing the argument in [EL93] which proves that if X is a smooth complex projective surface then the Seshadri constant of an ample line bundle on X at a very general point is ≥ 1. The key ingredient is to show that if $(C_t, x_t)_{t\in T}$ is a family of irreducible pointed curves on X where x_t moves and $mult_{x_t}(C_t) = m$ for every $t \in T$, then $(C_t^2) \geq m(m-1)$. This does not hold in positive characteristic: for example, one can consider a morphism $\pi\colon\ mapX.T.$ such that each fiber $C_t = \pi^{-1}(t)$ has a point x_t of multiplicity $m = 2$ (in this case $(C_t^2) = 0$).

However, in order to obtain the same lower bound for the Seshadri constant that holds in characteristic zero, it would be enough to prove the above lower bound for (C_t^2) under the assumption that the points x_t cover an open subset of X. In this case it is easy to see that $(C_t^2) \geq m$. However, in order to obtain a uniform lower bound for the Seshadri constant one would need an equality of the form $(C_t^2) \geq \alpha m^2$. At this point, it is not clear whether one can expect such a bound.

The group also investigated the possibility of constructing some interesting examples, specific to positive characteristic. More precisely, the idea would be to use quotients by suitable foliations and to compute Seshadri constants for the images of various curves under such quotient maps. However, the explicit construction of interesting such examples remains a project for the future.

Connectedness:

In characteristic zero, Kollár and Shokurov proved that if X is projective and $-(K_X + \Delta)$ is ample then the non kawamata log terminal locus of the pair (X, Δ) is connected.

In characteristic p, it is natural to try to prove that the non kawamata log terminal locus is connected when X is F-regular. Unfortunately this problem seems very hard and this problem was studied for only one day.

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