Mock modular forms in combinatorics and arithmetic geometry

organized by
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Workshop Summary

The primary goal of this workshop was to focus on some of the different areas of mathematics and physics that could benefit from the many recent advances in the area of mock modular forms. In the mornings there were one or two introductory lectures. On the first day, George Andrews gave an talk on various techniques and results in the theory of q-series. Don Zagier brought the participants up to speed on Nahm's conjecture, which gives a (deep) relation between the modularity of q-series and the Bloch group. On day two Wladimir Pribitkin gave an historic account of modular integrals, first introduced by Niebur in the late sixties, and Jan Bruinier explained the connection between mock theta functions and Borcherds type products. On Wednesday Sander Zwegers gave a talk on higher depth mock modular forms, a further generalization of the "classical" mock theta functions. On the last two days we saw some further applications of mock theta functions: Andreas Malmendier explained the connection between mock theta functions and the classification of manifolds, and Karl Mahlburg spoke about bounds on metastability thresholds and probabilities for generalized bootstrap percolation.

In the afternoons the participants broke up into discussion/working groups. Some of these groups were devoted to the subject of the morning talk, where the speaker would provide further details and the participants could ask more questions. We now give a summary of some of the main themes of discussion.

Nahm's conjecture: Nahm's conjecture involves the problem of understanding the overlap between the classes of q-hypergeometric functions and modular forms. The (conjectural) answer involves dilogarithms and the Bloch group. After Don Zagier gave an excellent introduction into the subject, several afternoons were spent on further discussions. These discussions were mainly brainstorming sessions where ideas were being thrown around. One possible way to attack the problem is to consider the asymptotic expansion of the q-hypergeometric function as q approaches a root of unity. In principle this technique could very well gave a proof of the conjecture in one direction: if a q-hypergeometric series is modular, then the corresponding element in the Bloch group is torsion. This approach has been worked out for the rank 1 case, but for larger values of r it becomes computationally very hard. Several ideas were discussed to get better asymptotics and how to make better use of them. In the other direction we have the problem of finding an actual identity once we know that a particular q-hypergeometric series is modular. Several techniques coming from the theory of q-series were discussed and also how they "translate" on the Bloch group side. While no actual results were obtained, several new ideas were given and this will be the start of more future research.

Algebraicity: Although there are natural algebraic structures within the classical theory of modular forms (e.g. singular moduli, Hecke eigenvalues, Fourier coefficients of newforms, etc.), we have come to understand that the situation is more complicated in the theory of harmonic Maass forms. In this regard, there were daily discussions on the algebraicity of mock modular forms of integer weight and half-integral weight, as well as discussions on the algebraicity of CM values of suitable harmonic Maass forms. In the context of harmonic Maass forms with integer weight, it is known (by work of Bruinier, Ono, and Rhoades), that mock modular forms associated to CM forms have algebraic coefficients. Luca Candelori gave a lecture on his thesis, where (in the case of weight 2) he refined this result to show that the mock modular forms are defined over the minimal possible field of definition. There were discussions on how to generalize this result to all integer weights. For half-integral weights, Bruinier spoke on a recent joint paper with Ono which equates the vanishing of derivatives of modular L-functions to the algebraicity of corresponding Fourier coefficients. Many discussions involved the problem of making this phenomenon more precise. Namely, obtain a direct formula for these algebraic coefficients in terms of the derivative of the L-function, or the heights of Heegner points, or periods. There was also many discussions on the algebraicity of harmonic Maass forms at CM points. Preliminary results were obtained in this direction.

Percolation: By work of Andrews, Holroyd, Liggett and Romik it turns out that mock theta functions are related to certain cellular automata models. From an automorphic point this corresponds to understanding products of Ramanujan's mock theta functions with classical weakly holomorphic forms. Karl Mahlburg gave an introduction in to the subject and described how different areas are connected here: number theory, combinatorics, probability, and cellular automata models. In the discussion following, particular asymptotic formulas and how the interplay between the areas might help here, were discussed.

In particular the question arose whether certain injections for partitions may help to understand certain probability events. This is still an ongoing discussion.

Over all, the workshop was very successful. Various applications and new ideas were communicated to a wider audience and progress was made in several directions.