Moduli spaces for algebraic dynamical systems

organized by
Laura DeMarco, Clay Petsche, and Joseph Silverman

Workshop Summary

A conference at AIM on
Moduli Spaces for Algebraic Dynamical Systems
September 27 to October 1, 2021
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The time has come,
The Walrus said,
To speak of many things,
Of Ends and Auts and moduli,
Of GIT and rings (of invariants)

Workshop Summary

This AIM workshop was devoted to the study of the geometry and arithmetic of moduli spaces associated to dynamical systems on algebraic varieties, with a particular emphasis on dynamical systems on projective space. As usual at AIM workshops, there were talks most mornings, and research working groups in the afternoons. The talks were designed to introduce participants to the primary topics and to some of the tools used in their study, plus some talks that discussed recent work. The main topics for the workshop were:

- The geometry of $\mathcal{M}_d^N$, the moduli space of degree d endomorphisms of $\mathbb{P}^N$.
- Arithmetic problems associated to rational points on $\mathcal{M}_d^N$.

There were four or five groups meeting during each work session (Tuesday to Thursday afternoons, and Friday moorning), with some of the groups meeting all four session, while others met fewer times as new topics became popular among the participants. Among the problems studied in the working groups are the following:

I: Critical Height Problem in Higher Dimensions

The critical height of a rational map $f : \mathbb{P}^1_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is the function

$$\hat{h}^{\text{crit}} : \mathcal{M}_d^1(\overline{\mathbb{Q}}) \longrightarrow [0,\infty), \quad \hat{h}^{\text{crit}}(f) = \sum_{c \in \text{Crit}(f)} \hat{h}_f(c).$$

It was proven (as an outgrowth of an earlier AIM workshop) that $\hat{h}^{\text{crit}}$ defines a Weil height on (most of) $\mathcal{M}_d^1$. The proof involves two components, which are connected by the multiplier map

$$\sigma^{(k)} : \mathcal{M}_d^1 \longrightarrow \mathbb{A}^K,$$

where $\sigma^{(k)}(f)$ is defined using the symmetric functions of the multipilers of the k-periodic points of f.

(I) Prove that $\hat{h}^{\mathrm{crit}}$ is equivalent to the *multiplier height*

$$\hat{h}^{\mathrm{mult}} : \mathcal{M}_d^1(\overline{\mathbb{Q}}) \longrightarrow [0,\infty), \quad \hat{h}^{\mathrm{mult}}(f) = h\bigl(\sigma^{(k)}(f)\bigr).$$

(II) Prove that the multiplier height is equivalent to a Weil height.

Step (II) is a consequence of a theorem of McMullen in this case. The problem considered by the working group was to generalize this result to endomorphisms of $\mathbb{P}^N$ for $N \geq 2$. In this case the critical locus of f is a codimension 1 subvariety C_f, and the critical height $\hat{h}^{\mathrm{crit}}(f) = \hat{h}_f(C_f)$ is the associated canonical height of a subvariety. The group concentrated on Step (I) for $N = 2$ and made some progress in understanding the local computations that would be required for a proof.

II: Shafarevich Locus Problem

An endomorphism $f \in \mathrm{End}_d^N(\overline{\mathbb{Q}})$ has good reduction at a prime $\mathfrak{p}$ if, after change of coordinates, its reduction modulo $\mathfrak{p}$ is still a morphism of degree d. The Shafarevich locus Shaf_d^N of $\mathcal{M}_d^N$ is the Zariski closure of the maps that have everywhere good reduction, i.e., that have good reduction at every prime. Some information was known about the dimension Shaf_d^1, and one of the organizers had rashly suggested that in general, the dimension Shaf_d^N should be significantly smaller than the dimension of $\mathcal{M}_d^N$. A working group showed that this suggestion was false, and that indeed one has

$$\liminf_{d\to\infty} \frac{\dim \mathrm{Shaf}_d^N}{\dim \mathcal{M}_d^N} \geq 1 - \frac{1}{N+1}.$$

III: Dynamical Szpiro Conjecture

A group investigated the possibility of formulating a dynamical analogue of Szpiro's discriminant-conductor conjecture for elliptic curves. One approach was to consider a notion of minimal discriminant that captures bad reduction of rational maps equipped with portrait structure, following the Shafarevich-type finiteness theorems of Silverman. The group also explored other dynamical notions of bad reduction involving critical orbits, as well as the possibility of bounding the critical height on the moduli space in terms of the places of bad reduction, an approach which would be analogous to Freys conjecture on the height of the j-invariant of an elliptic curve. The group also considered an analytic approach involving families of rational maps whose Berkovich Julia sets have finite hyperbolic diameter at all places.

IV: Geometry of $\mathcal{M}_d^N$

There were two groups that worked on different aspects of the geometry of the moduli space $\mathcal{M}_d^N$ and it covering spaces. They worked to develop tools to attack the following questions:

- Is that $\mathcal{M}_d^N$ a rational variety?
- What is the class of $\mathcal{M}_d^1$ in the Grothendieck ring of varieties?
- Let $\mathcal{M}_d^N[m]$ denote the moduli space of pairs (f, P), where P is a periodic point of (formal) period m for f. Is it true that $\mathcal{M}_2^1[m]$ is of general type for $m \geq 6$? (It is known that it is of general type for $m = 6$.) More generaly, is $\mathcal{M}_d^N[m]$ of general type for $m \geq C(d, N)$?
- Examine compactifications of $\mathcal{M}_d^N$ built from GIT and determine the boundary structure and applicability to dynamical problems
- Find bounds for the gonality of $\mathcal{M}_2^1[m]$, and more generally for the gonality of $\mathcal{M}_d^N[m]$.

It is known for $N = 1$ that $\mathcal{M}_d^1$ is rational, by a result of Levy. One group studied his proof and discussed variations to extend it to higher N. They also found an argument for computing the class of $\mathcal{M}_d^1$ in the Grothendieck ring of varieties, but later found some very recent work on the ArXiv carrying out a similar computation. For the problem of general type, a group focused on the case of $N = 1$ and $d = 2$ and studied the ramification locus of the projection from $\mathcal{M}_2^1[m]$ to $\mathcal{M}_2^1$. Finally, for the compactification problem, a group focused on the case of $N = 2$ and $d = 2$ and rediscovered earlier results in the literature before initiating a computational strategy to obtain data for higher degrees d.

IV: Coincident Dynamically Defined Fields

This group studied questions of the following general form: Let $f, g \in \mathrm{End}_d^N(K)$, and let K_f and K_g be "dynamically defined fields" related to f and g. If $K_f = K_g$, what can be said about the relation between f and g. A number of different types of dynamically defined fields were proposed, including the following:

- Let K_f be the field generated by all of the periodic (or preperiodic) points of f.
- Let K_f be the field generated by all of the multipliers of all of the periodic points of f.
- For a fixed m, let K_f be the field generated by all of the m-periodic points of f.
- Fix a point P and let $K_{f,P}$ be the field generated by the full backward orbit of P.
- Let K_f be the field generated by the full backward orbit of the critical points of f.

The group concentrated on the fields generated by all periodic points and by all multipliers. They focused on the simplest nontrivial setting, that of the quadratic polynomials $f(z) = z^2+c$. They found some recent results in the literature that addressed special cases or related problems, using both arithmetic and complex-analytic methods. For example, a conjecture of Milnor asked when all multipliers are integers (in $\mathbb{Q}$ or in some given number field), and special cases for maps in degree 2 were treated in the recent PhD thesis of Valentin Huguin.