Developments in moduli problems

organized by
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Workshop Summary

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Summary

This workshop brought together experts in different aspects of moduli theory in algebraic geometry. The main goal was to explore applications of recent developments in the foundations of moduli theory such as existence theorems for good moduli spaces and Θ-stratifications to a variety of moduli problems. The workshop identified common challenges and ideas arising from applying these techniques in different contexts.

Speakers

- Monday: Andres Fernandez Herrero and Yuchen Liu
- Tuesday: Tomás Gómez and Filippo Viviani
- Wednesday: Eloise Hamilton and Jacob Keller
- Thursday: Jochen Heinloth and Giovanni Inchiostro
- Friday: Daniel Halpern-Leistner, Chiara Damiolini, and David Rydh

Problems

Vector bundles on higher dimensional varieties.
On a projective variety X of dimension greater than one, the moduli space of semistable vector bundles is not compact. This space is usually compactified by parameterizing semistable torsion free sheaves. A working group investigated whether there is an alternative compactification parameterizing certain pairs (X', E) where $X' \to X$ is a birational modification and E is a vector bundle on X'. This would be analogous to Caporaso's compactification of the Jacobian of a nodal curve. Progress was made during the workshop. While the initial focus was on the dimension 2 case, it was realized that the same ideas would apply in all dimension. It was speculated that if one parameterizes pairs (X', E) such that the fibers of $X' \to X$ have slc singularities and X' satisfies Kollár's condition, then infinite dimensional GIT may succeed in constructing a projective compactification. A working group will continue to meet to continue this project.

Compactifications of M_g.
A working group investigated on how to construct birational models of $\bar{M}_g$ using 'beyond GIT' techniques. Let $\mathcal{M}$ be a suitable enlargement of the moduli stack of stable curves such as the stack of lci curves; there are other choices of enlargement possible and part of the challenge is to identify a good one. The $\mathbb{Q}$-line bundle $K + \alpha\delta$ on $\mathcal{M}$ defines a stability condition. The longterm and ambitious problem is to identify the semistable locus and

compare them to the moduli stacks appearing the Hassett–Keel program. One tractable problem discussed during the workshop was the show that for $\alpha > 9/11$, the semistable locus corresponds precisely to Deligne–Mumford stable curves. Partial progress was made by using slope inequalities and reductions of test configurations to nodal curves. A working group will continue to work on this project.

Moduli of pairs.

This project aims to study birational geometry of moduli space of curves using moduli space of surface pairs from KSBA or K-stability. A motivating example is in genus 3, where every canonical curve C is embedded in $\mathbb{P}^2$ as a quartic curve, and the KSBA/K-moduli spaces of $(\mathbb{P}^2, cC)$ provide an alternative description of the Hassett-Keel program for $\overline{M}_3$. During the workshop, a working group studied moduli of curves C on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (a, b) where $a < b$. In this case, the moduli spaces of $(\mathbb{P}^1 \times \mathbb{P}^1, cC)$ is defined using K-stability or KSBA if $c < \frac{1}{b}$ or $c > \frac{1}{a}$. Thus an interesting question is to construct new moduli spaces for the range $c \in [\frac{1}{b}, \frac{1}{a}]$. The working group made some progress in understanding the boundary cases when $c = \frac{1}{b}$ or $c = \frac{1}{a}$, in which a natural choice is the ample model of the CM line bundle. It is also related to constructing moduli spaces of KSBA/K-stable generalized pairs. A working group may continue to meet and work on this project.

SL.
$_n$*-bundles on nodal curves*

The original problem was, for a semisimple and simply connected group G and a projective curve C with arbitrary singularities, to find open substacks of $\mathrm{Bun}_G(C)$ that admit good moduli spaces. Ideally these good moduli spaces would be isomorphic to the Proj of the section ring of the inverse determinant of cohomology line bundle on $\mathrm{Bun}_G(C)$. When C is nodal, this section ring is the algebra of conformal blocks associated to C, and it would be wonderful to find a modular interpretation of its Proj. We have been focusing on the case where $G = \mathrm{SL}_n$ and C is a nodal curve, as that is a relatively simple but very important case. We believe the theta-semistable locus in $\mathrm{Bun}_G(C)$ does not have a good moduli space, because we found an unbounded family of theta-semistable SL_2 bundles on an irreducible nodal curve. The main issue is that there aren't enough destabilizing subbundles of a given bundle. To get around this, we are considering the enlarged moduli problem of pairs (C', E), where C' is a semistable model of a stable nodal curve C (potentially with orbifold structure), and E is an SL_n bundle on C'. The goal would now be to find an open substack of this larger stack with a good moduli space isomorphic to the Proj of the algebra of conformal blocks on C. A working group will continue to meet and approach this problem using the idea of infinite-dimensional GIT.

Stacky non-reductive GIT.

Recent work of Gergely Berczi, Brent Doran, Frances Kirwan, and others extends geometric invariant theory (GIT) to quotients of non-reductive groups. The longterm project seeks to offer a stack-theoretic treatment of non-reductive GIT, to extend techniques of 'beyond GIT' to this setting, and to apply the theory to concrete examples. David Rydh has recently made partial progress to this goal and presented some of his work at this workshop. David introduced an extension of the notion a good moduli space called a 'non-reductive moduli space' characterizing non-reductive quotients and conjectured during the workshop that an algebraic stack admits a separated non-reductive moduli space if and only if the stack satisfies certain generalizations of the notions of Θ-reductivity and S-completeness. A working group met to understand David's work and discussed how his results might be extended. We will continue to meet to work on this problem.

Projectivity via infinite dimensional GIT.
A working group brainstormed stack-theoretic techniques to prove ampleness of tautological line bundles on moduli spaces of sheaves and G-bundles. There was a promising technique proposed using infinite dimensional GIT, but a detailed analysis revealed that it wouldn't work due to sign issues.

Weighted hypersurfaces and K-stability.
The motivation of this project is to study compact moduli spaces for weighted hypersurfaces. It is a testing ground for various different theories, such as non-reductive GIT (e.g. work of Bunnett) and K-stability (e.g. work of Ascher-DeVleming-Liu). Thus it is an interesting question to compare these two approaches. At the workshop, some examples are discussed such as curves on $\mathbb{P}(1, 1, d)$ and certain log del Pezzo surfaces as weighted hypersurfaces. A working group will continue to meet and work on this project.

Local structure of algebraic stacks in characteristic p.

A working group met on Friday to discuss extensions of the local structure theorem of Alper-Hall-Rydh to algebraic stacks around points with reductive (rather than linearly reductive) stabilizer groups. While a general theorem seems out of reach at the moment, several tractible problems were identified and discussed. A working group may meet in the future to continue the discussions.