Carleson theorems and multilinear operators: Open problems

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Question 1 (Thiele)**.** *Let $u : \mathbb{R} \to \mathbb{R}$ be a measurable function. Define the maximal operator along the planar vector field $(1, u)$ by*

$$M_u f(x, y) := \sup_{\epsilon>0} \left| \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x - t, y - u(x)t) dt \right|. \tag{0.1}$$

Does M_u satisfy any L^p bound for certain $p < \infty$?

Question 2 (Christ)**.** *On $\mathbb{R}^d$, let $E \subset \mathbb{R}^d$ with $|E| = 1$. Given $q > 2$, describe the set E that maximises the quantity $\|\widehat{\mathbb{1}_E}\|_q$.*

Certain partial progress has been made.

Theorem 0.1 (Christ [3])**.** *1. For any $q > 2$, the extremizing set exists.*

2. For any dimension d, for any sufficiently large q which is also sufficiently close to $2\mathbb{N}$, a set E is a extremizer iff E is an ellipsoid.

3. If $d = 1$, then for any q close to $2\mathbb{N}$, a set E is a extremizer iff E is an ellipsoid.

4. If $d = 2$, then for any q close to 4, a set E is a extremizer iff E is an ellipsoid.

Question 3 (Christ)**.** *Let B be the unit ball in $\mathbb{R}^3$. Let N be a positive integer. Let $\{V_j : 1 \leq j \leq N\}$ be N different light cones in $\mathbb{R}^3$. Prove that*

$$\left| \int_B e^{i\lambda x_3^2} \prod_{j=1}^{N} f_j(x \cdot v_j) dx \right| \lesssim \lambda^{-\epsilon} \prod_{j=1}^{N} \|f_j\|_\infty, \tag{0.2}$$

for certain positive ϵ, where $v_j \in V_j$. If possible, find the optimal ϵ.

So far (0.2) has only been proved for $N \leq 5$, see [4].

Question 4 (Bennett)**.** *Suppose we are in $\mathbb{R}^4$. Let $\epsilon > 0$. Suppose that $\mathbb{T}_1$, $\mathbb{T}_2$ and $\mathbb{T}_3$ are three transversal families of δ-tubes (short sides δ and long side 1) such that for each $j \in \{1, 2, 3\}$, $\{e(T_j) : T_j \in \mathbb{T}_j\}$ forms a δ-separated subset of $\mathbb{S}^3$. If $q \geq \frac{4}{3}$ and $\frac{1}{p} + \frac{3}{q} \leq 3$, then there exists a constant $C_\epsilon > 0$ such that*

$$\left\| \prod_{j=1}^{3} \left(\sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \right) \right\|_{L^{q/3}(\mathbb{R}^4)} \leq C_\epsilon \prod_{j=1}^{3} \delta^{\frac{4}{q} - \frac{3}{p'} - \epsilon} (\#\mathbb{T}_j)^{1/p}. \tag{0.3}$$

Here $e(T) \in \mathbb{S}^3$ denotes the direction of the long side of a tube T.

Question 5 (Di Plinio)**.** *Let $N \in \mathbb{N}$. Given a collection of N directions $\{v_j \in \mathbb{S}^1 : 1 \leq j \leq N\}$. Does it hold true that*

$$\| \sup_{j \in \{1,2,\ldots,N\}} |H_{v_j} f| \|_{L^{2,\infty}(\mathbb{R}^2)} \lesssim \sqrt{\log N} \|f\|_2? \tag{0.4}$$

Here $H_{v_j} f$ denotes the Hilbert transform along the given direction v_j, namely

$$H_{v_j} f(x) := \int_{\mathbb{R}} f(x - t v_j) \frac{dt}{t}. \tag{0.5}$$

The maximal variant of the estimate (0.4) has been proved by Katz [8]. Moreover, (0.4) has also been verified for sets of two "extreme" structures: the lacunary set and the Vargas set. One typical example of the Vargas set is the set of uniformly distributed directions. See Demeter [5], Demeter and Di Plinio [6].

Question 6 (Street)**.** *Prove*

$$\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x) g(y) h(x+y) \frac{1}{det(x,y)} dx dy \right| \lesssim \|f\|_p \|g\|_q \|h\|_r, \tag{0.6}$$

for certain p, q and r.

This question has a quite satisfactory answer. See Gressman et al. [7].

Question 7 (Krause)**.** *On the plane $\mathbb{R}^2$, let $R : \mathbb{S}^1 \to \mathbb{S}^1$ be the rotation by $\pi/3$. Prove*

$$\left\| \sup_{t\in\mathbb{R}^+} \left| \int_{\mathbb{S}^1} f(x - tw)g(x - tR(w))d\sigma(w) \right| \right\|_r \lesssim \|f\|_p \|g\|_q, \tag{0.7}$$

for certain p, q and r.

During the workshop, this has been shown to be equivalent to

$$\left\| \sup_{t\in\mathbb{R}^+} \left| \int_{\mathbb{S}^1} f(x - tw)g(x + tw)d\sigma(w) \right| \right\|_r \lesssim \|f\|_p \|g\|_q. \tag{0.8}$$

Question 8 (Anderson, Pierce)**.** *Generalise Stein and Wainger's polynomial Carleson's theorem to the discrete setting, namely to prove*

$$\left\| \sup_{\lambda} \left| \sum_{m\in\mathbb{Z}} f(n - m) \frac{e^{i\lambda m^2}}{m} \right| \right\|_2 \lesssim \|f\|_2. \tag{0.9}$$

Let $\Lambda \subset [0, 1]$. Define

$$\sup_{\lambda\in\Lambda} \left| \sum_{m\in\mathbb{Z}} f(n - m) \frac{e^{i\lambda m^2}}{m} \right|. \tag{0.10}$$

A sufficient condition has been given on the set Λ, to guarantee the l^2 boundedness of (0.10). See Krause and Lacey [9].

Question 9 (Li)**.** *Let $d \geq 3$. For $p \geq 2(d + 1)$, prove*

$$\| \sum_{n=1}^{N} a_n e^{2\pi i n^d t} e^{2\pi i n\cdot x} \|_{L^p(\mathbb{T}^2)} \lesssim N^{\frac{1}{2} - \frac{d+1}{p} + \epsilon} (\sum_{n=1}^{N} |a_n|^2)^{1/2}. \tag{0.11}$$

This is related to Waring's problem.

Question 10 (Bez)**.** *Let S_1, S_2 and S_3 be transversal patches of the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$. Let σ_1, σ_2 and σ_3 be the surface measure separately. Determine the full range of exponents $p, q > 0$ such that the multi-linear singular convolution estimate*

$$\|g_1 d\sigma_1 * g_2 d\sigma_2 * g_3 d\sigma_3\|_q \lesssim \prod_{j=1}^{3} \|g_j\|_p \tag{0.12}$$

holds.

Question 11 (Muscalu). *Let $K : \mathbb{R}^2 \to \mathbb{R}$ be a function such that*

$$|\partial^\alpha \hat{K}(\xi)| \lesssim \frac{1}{|\xi|^{|\alpha|}}, \forall \xi \in \mathbb{R}^2 \setminus \{0\}, \tag{0.13}$$

for sufficiently many multi-indices α. Generalise Stein and Wainger's polynomial Carleson's theorem to the multi-linear setting. For example, to prove

$$\| \sup_{\lambda \in \mathbb{R}} | \int_{\mathbb{R}^2} f(x-t) g(x-s) K(t,s) e^{i\lambda s^2 t^2} dt ds | \|_2 \lesssim \|f\|_4 \|g\|_4. \tag{0.14}$$

The multi-parameter Carleson's theorem has been proved by Li and Muscalu [11]: Let K be given as in (0.13). Define

$$C_2(f,g)(x) := \sup_{N_1, N_2} \left| \int_{\mathbb{R}^2} \hat{K}(\xi_1 - N_1, \xi_2, N_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) d\xi_1 d\xi_2 \right|, \tag{0.15}$$

then

$$\|C_2(f_1, f_2)\|_2 \lesssim \|f_1\|_4 \|f_2\|_4. \tag{0.16}$$

Question 12 (Guo). *To prove that there exists a universal constant $C > 0$ such that $\forall \epsilon \in (0, 1/2)$, it holds that*

$$\| \sup_{\lambda \in \mathbb{R}} \int_{\mathbb{R}} f(x-t) e^{i\lambda |t|^\epsilon} \frac{dt}{t} \|_2 \le C \|f\|_2. \tag{0.17}$$

Question 13 (Carbery). *On $\mathbb{R}^n$, it is a big open problem whether*

$$\left\| \sup_R \left| \int_{|\xi| \le R} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \right| \right\|_2 \lesssim \|f\|_2. \tag{0.18}$$

How about

$$\left\| \sup_R \left| f * \left(\frac{e^{i|x|}}{|x|^{\frac{n+1}{2}}} \cdot \chi_{\{|x| \le R\}} \right) \right| \right\|_2 \lesssim \|f\|_2 ? \tag{0.19}$$

For detailed discussions, see Carbery et al. [2].

Question 14 (Iliopoulou)**.** *In $\mathbb{R}^n$, let $\mathcal{T}_i, i \in \{1, 2, ..., n\}$ be a collection of tubes with width one and infinity length. We know that*

$$\int \left[\left(\sum_{T_1 \in \mathcal{T}_1} \chi_{T_1} \right) \cdots \left(\sum_{T_n \in \mathcal{T}_n} \chi_{T_n} \right) w(T_1) \wedge ... \wedge w(T_n) \right]^{\frac{1}{n-1}} \leq C_n \prod_{i=1}^{n} (\#\mathcal{T}_i)^{\frac{1}{n-1}}, \tag{0.20}$$

where $w(T_i)$ is the unit vector parallel to the long side of the tube T_i. Could we prove (0.20) *with $C_n = 1$?*

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