Carleson theorems and multilinear operators: 
Open problems

Shaoming Guo

**Question 1** (Thiele). Let \( u : \mathbb{R} \to \mathbb{R} \) be a measurable function. Define the maximal operator along the planar vector field \((1, u)\) by

\[
M_u f(x, y) := \sup_{\epsilon > 0} \left| \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x - t, y - u(x)t) dt \right|.
\]

(0.1)

Does \( M_u \) satisfy any \( L^p \) bound for certain \( p < \infty \)?

**Question 2** (Christ). On \( \mathbb{R}^d \), let \( E \subset \mathbb{R}^d \) with \( |E| = 1 \). Given \( q > 2 \), describe the set \( E \) that maximises the quantity \( \| \mathbf{1}_E \|_q \).

Certain partial progress has been made.

**Theorem 0.1** (Christ [3]).

1. For any \( q > 2 \), the extremizing set exists.

2. For any dimension \( d \), for any sufficiently large \( q \) which is also sufficiently close to \( 2N \), a set \( E \) is a extremizer iff \( E \) is an ellipsoid.

3. If \( d = 1 \), then for any \( q \) close to \( 2N \), a set \( E \) is a extremizer iff \( E \) is an ellipsoid.

4. If \( d = 2 \), then for any \( q \) close to \( 4 \), a set \( E \) is a extremizer iff \( E \) is an ellipsoid.

**Question 3** (Christ). Let \( B \) be the unit ball in \( \mathbb{R}^3 \). Let \( N \) be a positive integer. Let \( \{V_j : 1 \leq j \leq N\} \) be \( N \) different light cones in \( \mathbb{R}^3 \). Prove that

\[
\left| \int_B e^{i\lambda x} \prod_{j=1}^N f_j(x \cdot v_j) dx \right| \lesssim \lambda^{-\epsilon} \prod_{j=1}^N \|f_j\|_\infty,
\]

(0.2)

for certain positive \( \epsilon \), where \( v_j \in V_j \). If possible, find the optimal \( \epsilon \).
So far (0.2) has only been proved for $N \leq 5$, see [4].

**Question 4** (Bennett). *Suppose we are in $\mathbb{R}^4$. Let $\epsilon > 0$. Suppose that $T_1$, $T_2$ and $T_3$ are three transversal families of $\delta$-tubes (short sides $\delta$ and long side 1) such that for each $j \in \{1, 2, 3\}$, $\{e(T_j) : T_j \in T_j\}$ forms a $\delta$-separated subset of $S^3$. If $q \geq \frac{4}{3}$ and $\frac{4}{p} + \frac{3}{q} \leq 3$, then there exists a constant $C_\epsilon > 0$ such that

$$
\left\| \prod_{j=1}^{3} \left( \sum_{T_j \in T_j} \chi_{T_j} \right) \right\|_{L^{3/q}(\mathbb{R}^4)} \leq C_\epsilon \prod_{j=1}^{3} \delta^{\frac{4}{q} - \frac{3}{p} - \epsilon} (\#T_j)^{1/p}.
$$

(0.3)

Here $e(T) \in S^3$ denotes the direction of the long side of a tube $T$.

**Question 5** (Di Plinio). *Let $N \in \mathbb{N}$. Given a collection of $N$ directions $\{v_j \in S^1 : 1 \leq j \leq N\}$. Does it hold true that

$$
\left\| \sup_{j \in \{1, 2, \ldots, N\}} |H_{v_j}f| \right\|_{L^{2, \infty}(\mathbb{R}^2)} \lesssim \sqrt{\log N} \|f\|_2?
$$

(0.4)

Here $H_{v_j}f$ denotes the Hilbert transform along the given direction $v_j$, namely

$$
H_{v_j}f(x) := \int_{\mathbb{R}} f(x - tv_j) \frac{dt}{t}.
$$

The maximal variant of the estimate (0.4) has been proved by Katz [8]. Moreover, (0.4) has also been verified for sets of two “extreme” structures: the lacunary set and the Vargas set. One typical example of the Vargas set is the set of uniformly distributed directions. See Demeter [5], Demeter and Di Plinio [6].

**Question 6** (Street). *Prove

$$
\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x)g(y)h(x + y) \frac{1}{\text{det}(x, y)} \, dx \, dy \right| \lesssim \|f\|_p \|g\|_q \|h\|_r,
$$

(0.6)

for certain $p, q$ and $r$.

This question has a quite satisfactory answer. See Gressman et al. [7].
**Question 7** (Krause). On the plane $\mathbb{R}^2$, let $R : S^1 \to S^1$ be the rotation by $\pi/3$. Prove

$$
\left\| \sup_{t \in \mathbb{R}^+} \left| \int_{S^1} f(x - tw)g(x - tR(w))d\sigma(w) \right| \right\|_r \lesssim \|f\|_p \|g\|_q, \quad (0.7)
$$

for certain $p, q$ and $r$.

During the workshop, this has been shown to be equivalent to

$$
\left\| \sup_{t \in \mathbb{R}^+} \left| \int_{S^1} f(x - tw)g(x + tw)d\sigma(w) \right| \right\|_r \lesssim \|f\|_p \|g\|_q. \quad (0.8)
$$

**Question 8** (Anderson, Pierce). Generalise Stein and Wainger’s polynomial Carleson’s theorem to the discrete setting, namely to prove

$$
\left\| \sup_{\lambda \in \Lambda} \left| \sum_{m \in \mathbb{Z}} f(n - m) e^{i\lambda m^2} \right| \right\|_2 \lesssim \|f\|_2. \quad (0.9)
$$

Let $\Lambda \subset [0, 1]$. Define

$$
\sup_{\lambda \in \Lambda} \left| \sum_{m \in \mathbb{Z}} f(n - m) e^{i\lambda m^2} \right|. \quad (0.10)
$$

A sufficient condition has been given on the set $\Lambda$, to guarantee the $l^2$ boundedness of (0.10). See Krause and Lacey [9].

**Question 9** (Li). Let $d \geq 3$. For $p \geq 2(d + 1)$, prove

$$
\left\| \sum_{n=1}^{N} a_n e^{2\pi i n d t} e^{2\pi i n - x} \right\|_{L^p(T^d)} \lesssim N^{1 - \frac{d+1}{p} + \frac{1}{2}} \left( \sum_{n=1}^{N} |a_n|^2 \right)^{1/2}. \quad (0.11)
$$

This is related to Waring’s problem.

**Question 10** (Bez). Let $S_1$, $S_2$ and $S_3$ be transversal patches of the unit sphere $S^3$ in $\mathbb{R}^4$. Let $\sigma_1, \sigma_2$ and $\sigma_3$ be the surface measure separately. Determine the full range of exponents $p, q > 0$ such that the multi-linear singular convolution estimate

$$
\|g_1 \sigma_1 * g_2 \sigma_2 * g_3 \sigma_3\|_q \lesssim \prod_{j=1}^{3} \|g_j\|_p \quad (0.12)
$$

holds.
Question 11 (Muscalu). Let $K : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function such that

$$\left| \partial^\alpha \hat{K}(\xi) \right| \lesssim \frac{1}{|\xi|^{|\alpha|}}, \forall \xi \in \mathbb{R}^2 \setminus \{0\},$$

(0.13)

for sufficiently many multi-indices $\alpha$. Generalise Stein and Wainger’s polynomial Carleson’s theorem to the multi-linear setting. For example, to prove

$$\| \sup_{\lambda \in \mathbb{R}} \left| \int_{\mathbb{R}^2} f(x-t)g(x-s)K(t,s)e^{i\lambda s^2 t^2} dt ds \right|_2 \lesssim \|f\|_4 \|g\|_4.$$  

(0.14)

The multi-parameter Carleson’s theorem has been proved by Li and Muscalu [11]: Let $K$ be given as in (0.13). Define

$$C_2(f,g)(x) := \sup_{N_1,N_2} \left| \int_{\mathbb{R}^2} \hat{K}(\xi_1 - N_1, \xi_2, N_2) \hat{f_1}(\xi_1) \hat{f_2}(\xi_2) d\xi_1 d\xi_2 \right|,$$

(0.15)

then

$$\|C_2(f_1, f_2)\|_2 \lesssim \|f_1\|_4 \|f_2\|_4.$$  

(0.16)

Question 12 (Guo). To prove that there exists a universal constant $C > 0$ such that $\forall \epsilon \in (0, 1/2)$, it holds that

$$\| \sup_{\lambda \in \mathbb{R}} \int_{\mathbb{R}} f(x-t)e^{i\lambda |t|} \frac{dt}{t} \|_2 \leq C \|f\|_2.$$  

(0.17)

Question 13 (Carbery). On $\mathbb{R}^n$, it is a big open problem whether

$$\left\| \sup_{R} \left| \int_{|\xi| \leq R} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \right| \right\|_2 \lesssim \|f\|_2.$$  

(0.18)

How about

$$\left\| \sup_{R} \left| f * \left( \frac{e^{i|x|}}{|x|^{n+1}} \cdot \chi_{\{|x| \leq R\}} \right) \right| \right\|_2 \lesssim \|f\|_2?$$  

(0.19)

For detailed discussions, see Carbery et al. [2].
Question 14 (Iliopoulos). In $\mathbb{R}^n$, let $T_i, i \in \{1, 2, ..., n\}$ be a collection of tubes with width one and infinity length. We know that

$$\int \left[ \left( \sum_{T_1 \in T_1} \chi_{T_1} \right) \ldots \left( \sum_{T_n \in T_n} \chi_{T_n} \right) w(T_1) \wedge \ldots \wedge w(T_n) \right]^{\frac{1}{n-1}} \leq C_n \prod_{i=1}^{n} (\# T_i)^{\frac{1}{n-1}},$$

where $w(T_i)$ is the unit vector parallel to the long side of the tube $T_i$. Could we prove (0.20) with $C_n = 1$?

References


5