Dynamics of multiple maps

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Workshop Summary

Workshop Overview

In classical arithmetic dynamics, we consider the iteration of a single self-morphism $f : X \to X$ of a variety defined over a field of arithmetic interest — typically a number field or the function field of a curve. Over the past decade, a number of exciting results have been proved about the interaction between two or more self-morphisms of the same variety. We call this "dynamics of multiple maps", and our workshop was entirely devoted to a deeper look at results and conjectures in this area that might drive arithmetic dynamics in new directions.

The problems discussed during the workshop fell into four broad classes:

- The arithmetic of orbits for semigroup and stochastic dynamical systems;
- The arithmetic of orbits for correspondences;
- Finite orbits for non-commuting K3-surface automorphisms; and
- Common preperiodic points of multiple maps.

We close this overview with a description of each of these classes, and then we summarize the progress made in each of the working groups. Throughout, we write k for a field and $\bar{k}$ for its algebraic closure.

The arithmetic of orbits for semigroup and stochastic dynamical systems.
Let $S = \{f_0, f_1, \ldots, f_n\} \subset k(z)$ be a set of rational functions. Write G_S for the semigroup generated by S under composition:

$$G_S = \{g_1 \circ g_2 \circ \cdots \circ g_n : n \geq 1, g_i \in S\}.$$

The behavior of points in $\mathbb{P}^1(\bar{k})$ under elements of G_S is a natural extension to the theory of the dynamics of a single rational map. A genuinely new phenomenon arises if we equip S with a probability measure ν. We now consider the composition $g_1 \circ g_2 \circ \cdots \circ g_n$ as appearing with probability $\nu(g_1)\nu(g_2)\cdots\nu(g_n)$, and we refer to the pair (S, ν) as a stochastic dynamical system. In this context we can ask questions about the expected value of standard dynamical objects like canonical heights, arboreal representations, Julia sets, and equilibrium measures for the stochastic dynamical system (S, ν).

During the workshop, Wade Hindes, John Doyle, Vivian Healey, and Tom Scanlon gave talks related to this theme. See sections , , and for summaries of related working groups.

The arithmetic of orbits for correspondences.

Let $Y_{/k}$ be a variety, and let $X_{/k}$ be a finite disjoint union of varieties. Write $C : X \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} Y$ for a correspondence on Y — i.e., an ordered pair of finite morphisms $X \to Y$. The correspondence C defines a natural multi-valued map on the points $Y(\bar{k})$, with $y \mapsto y'$ if and only if there is $x \in X$ such that $\pi_1(x) = y$ and $\pi_2(x) = y'$. Iterating this map corresponds to walking around the "orbit graph" of C: the directed graph with edges $y \to y'$ for all $y \in Y(\bar{k})$ and all $y' \in \pi_2(\pi^{-1}(y))$. When π_1 is an isomorphism, this is a reformulation of single-map dynamics. Semigroup dynamics can also be fit into this framework by choosing X to be a finite union of copies of Y and π_1 an isomorphism on each component of X.

Certain natural notions from the single-map setting immediately become more interesting here. For example, if there is a finitely supported path in the orbit graph starting at $y \in Y(\bar{k})$, then we say then we say y is *constrained*. If the set of points in the orbit graph that are reachable by y is finite, then we say y is *fully constrained*. If the correspondence is single-valued these two notions are equivalent to y being preperiodic, however they are typically very different in general.

This area of study was new to many of the partipicants. Prior to the workshop, the organizers were only able to locate a few papers on the dynamics of correspondences over $\mathbb{C}$ and a couple of papers on height theory (by one of the organizers). This suggests there is ample room for rapid progress extending ideas from the theory of single-map dynamics.

The topic was introduced with a talk by Xander Faber on dyamics of correspondences, and the theme was revisited in talks Rafe Jones and Rob Benedetto. See sections , , , and for related problems.

Finite orbits for non-commuting K3-surface automorphisms.
K3-surfaces are another setting in which to study the dynamical interactions between multiple maps. For example, a smooth k-subvariety $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ cut out by a $(2, 2, 2)$-form is known as a "Wehler K3-surface". The projection $p_i \colon S \to \mathbb{P}^1 \times \mathbb{P}^1$ that drops the i-th coordinate is 2-to-1, and swapping the elements in the fiber of p_i gives rise to an automorphism $\sigma_i \in \mathrm{Aut}(S)$. Write $\mathcal{A}$ for the subgroup of $\mathrm{Aut}(S)$ generated by $\sigma_1, \sigma_2, \sigma_3$. Wehler showed that $\mathcal{A} \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, a free group on three generators. Write $S(k)_{\mathrm{fin}}$ for the set of points $P \in S(k)$ such that the orbit $\mathcal{A}.P$ is finite. Examples of K3-surfaces over the complex numbers with $S(\mathbb{C})_{\mathrm{fin}} \neq \varnothing$ are likely to be quite special.

Joe Silverman gave an introduction to dynamical questions related to non-commuting maps on surfaces. See Section for a description of the progress a working group made on this topic.

Common preperiodic points of multiple maps.
Write $\mathrm{Rat}_d(\mathbb{C})$ for the parameter space of morphisms $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree d. For such a morphism f, write $\mathrm{Preper}(f)$ for the set of (complex) preperiodic points of f. If $f, g \in \mathrm{Rat}_d$ have distinct Julia sets, then it is known that $\mathrm{Preper}(f) \cap \mathrm{Preper}(g)$ is a finite set. Moreover, it is believed that there is a uniform bound on these intersections:

$$C_d := \sup_{\substack{f,g \in \mathrm{Rat}_d(\mathbb{C}) \\ \mathcal{J}_f \neq \mathcal{J}_g}} |\mathrm{Preper}(f) \cap \mathrm{Preper}(g)| < \infty.$$

One would like to understand the value of C_d as a function of d, even in dynamically interesting subvarieties of $\mathrm{Rat}_d(\mathbb{C})$. It would also be interesting to know whether one expects this kind of uniformity for correspondences of fixed bidegree.

Trevor Hyde gave a talk on "dynamical configurations", which shed light on what such a uniform bound must look like. See sections and for descriptions of progress that working groups made on related problems.

Reports from Working Groups

Eight working groups met at least once over the course of the week, and all of them left AIM with the intention to continue collaborating. In addition, one problem was proposed that did not achieve critical mass (section), but which the organizers would like to highlight as being of future interest. Below are descriptions of the problems these groups studied.

Local to Global.

Let f be an endomorphism on $\mathbb{P}^N_{/\mathbb{Q}}$ with $\deg(f) \geq 2$. Assume that f has good reduction at a rational prime p, and let $\tilde{f}$ denote its reduction modulo p. For a point $\alpha \in \mathbb{P}^N(\mathbb{Q})$, information about the reduced orbit $\mathcal{O}_{\tilde{f}}(\tilde{\alpha})$ can be used to infer information about the original orbit $\mathcal{O}_f(\alpha)$.

Consider the maps $f_1, \ldots, f_s : \mathbb{P}^N \to \mathbb{P}^N$ and let $G = \langle f_1, \ldots, f_s \rangle$ be the semi-group they generate. The goal of this group was to determine how much information about the semigroup orbit of a point $\alpha \in \mathbb{P}^N(\mathbb{Q})$ can be recovered from the behavior of its orbit under the reduced semigroup modulo primes.

They proved that if the cardinality of the reduced semigroup orbit $\tilde{O}_G(\alpha)$ is uniformly bounded for infinitely primes p, then α is fully constrained under the entire semigroup G. They were able to prove this result using two different techniques: a model-theoretic technique using ultra-filters, and a more traditional algebraic argument.

A further objective was to extract more precise information about the possible periods of α under the action of G. In particular, they hoped to recover the "semigroup portrait" of α under G. Several instructive examples were created.

Path Heights.

Let $S = \{f_0, f_1, \ldots, f_n\} \subset k(z)$ be a set of rational functions. Let P be γ be an infinite sequence of elements of S and h_γ the canonical height associated to this path. For a generic base point $P \in \mathbb{P}^1(k)$, $h_\gamma(P)$ is the *path height* of γ. This group's work focused on understanding how the distribution of path heights at generic base points of $\mathbb{P}^1$ relate to properties of the underlying semigroup. They would eventually like to prove that the map $\gamma \to h_\gamma(P)$ is injective in many situations.

This group made progress on some related questions. They found a new expression for the expected value of the path height at P (using a limit involving the weighted geometric mean of the degrees of the maps in S); they proved a stochastic version of the Ping-Pong Lemma; and they have shown how the distribution of path heights at P detects the constants in the asymptotic growth rate of the number of points of bounded height in the semigroup orbit of P.

Dynamical Mordell-Lang.

The dynamical Mordell-Lang conjecture predicts that if $f : X \to X$ is an algebraic dymamical system over a field k of characteristic zero, $Y \subseteq X$ is an algebraic subvariety of

X, and $P \in X(k)$ is any k-rational point, then the set

$$E(X, f, Y, P) := \{n \in \mathbb{N} : f^{\circ n}(P) \in Y(K)\}$$

is a finite union of arithmetic progressions (possibly including degenerate arithmetic progressions consisting of a single point). While the conjecture has been proven in some cases, it remains open in full generality.

The Mordell-Lang problem for multiple maps seeks a characterization of the return sets

$$E(X, f_1, \dots, f_n, Y, P) := \{\gamma \in \langle f_1, \dots, f_n\rangle : \gamma(P) \in Y(k)\}$$

where $f_j : X \to X$ for $1 \leq j \leq n$ are endomorphisms of some given algebraic variety X over the field k, $Y \subseteq X$ is a subvariety, $P \in Y(k)$ is k-rational point, and $\langle f_1, \dots, f_n\rangle$ denotes the semigroup of regular maps generated by $\{f_1, \dots, f_n\}$. Variants of this problem have been considered in which the f_j's are assumed to commute or by looking instead at

$$\widetilde{E}(X, f_1, \dots, f_n, Y, P) := \{(\ell_1, \dots, \ell_n) \in \mathbb{N}^n : f_1^{\circ \ell_1} \circ \cdots \circ f_n^{\circ \ell_n}(P) \in Y(k)\}.$$

Reducing to the case that k is a number field, one sees that the set $E(X, f_1, \dots, f_n, Y, P)$ is primitive recursive, but, such sets can be complicated. It is known that every subset S of $\mathbb{N}^n$ defined by finitely many polynomial-exponential equations defined over the ring of integers of a number field may be realized as $\widetilde{E}(X, f_1, \dots, f_n, Y, P)$ for some *pairwise commuting* maps $f_j : X \to X$.

This proposed group, which did not meet during the AIM workshop, was tasked with

- constructing examples of systems of commuting maps for which $\widetilde{E}(X, f_1, \dots, f_n, Y, P)$ is not defined by polynomial-exponential equations,
- bringing the potential complexity of such return sets from primitive recursive to a simpler class, and
- finding reasonable geometric restrictions so that the sets $E(X, f_1, \dots, f_n, Y, P)$ could be expected to have a simpler form, e.g., as a finite union of translates of subsemigroups.

Arboreal Representations.

The aim of this group was to study arboreal-type questions in the correspondence setting. For background, let $f : X \to X$ be a finite self-morphism of a variety $X_{/k}$, and let $a \in X(k)$. Write $k_n := k\big(f^{-n}(a)\big)$ for the field of definition of the n-th pre-images of a, and write $k_\infty = \bigcup_{n \geq 1} k_n$. The arboreal Galois group for this pair (f, a) is $\mathrm{Gal}(k_\infty/k)$, and its natural action on the tree of pre-images of a is known as an arboreal representation. Understanding the image of $\mathrm{Gal}(k_\infty/k)$ inside the full automorphism group of this tree is the key problem in the study of arboreal representations. Much of the structure of these representations is controlled by the discriminants of the relative extensions k_{n+1}/k_n. In the case where $X = \mathbb{P}^1$ and f is a polynomial map, one can write down a formula for these discriminants.

Now switch to the correspondence setting. Write $C = D = \mathbb{P}^1$, and let $\phi : C \underset{f_2}{\overset{f_1}{\rightrightarrows}} \mathbb{P}^1$ and $\psi : D \underset{g_2}{\overset{g_1}{\rightrightarrows}} \mathbb{P}^1$ be two correspondences on $\mathbb{P}^1$, where f_1, g_1, f_2, g_2 are polynomials defined over

k. Diagrammatically, this is given by

$[rowsep = 3em]CD\mathbb{P}^1\mathbb{P}^1\mathbb{P}^1["f_1", from = 1{-}2, to = 2{-}1, swap]["f_2", from = 1{-}2, to = 2{-}3]["g_1", from =$

Let $a \in \mathbb{P}^1(k)$. The "correspondence pre-image" of a is the set

$$A = f_1(f_2^{-1}(g_1(g_2^{-1}(a)))).$$

The first goal of this group was to give a discriminant formula for the field extension $k(A)/k$. They also generated several classes of examples to explore and work out in detail.

p-adic dynamics.

This group studied k-rational constrained points for correspondences and semigroups, for k a number field. Aside from carrying out some basic computations to see how one may generate such points, they thought about whether key uniform boundedness results concerning k-rational preperiodic points of polynomial maps would carry over to k-rational constrained points of correspondences or semigroups. In particular, they hoped to prove that given a polynomial correspondence on $\mathbb{P}^1$ of bidegree (d_1, d_2) with $d_2 > d_1$, there are at most $Cs\log(s)$ k-rational constrained points, where s is the number of places of bad reduction of the correspondence, and C is some constant depending only on d and $[k : \mathbb{Q}]$. This would be the direct analogue of a well-known result of Benedetto for polynomial maps on $\mathbb{P}^1$.

Their conclusion was that the arguments in the polynomial case run into obstacles in this adapted setting, and that (outside of special cases like "unicritical" polynomial correspondences) these seem difficult to overcome. They did, however, generate some seemingly new notions of local canonical heights for semigroups and correspondences that would be interesting to better understand, both from a potential-theoretic and arithmetic perspective.

Complex Correspondence Dynamics.

Fix coprime positive integers p, q. For a complex number c , the "unicritical correspondence" $z \mapsto z^{p/q} + c$ is formally given by $\mathcal{F}_c : \mathbb{P}^1 \overset{z^q}{\underset{z^p+c}{\rightrightarrows}} \mathbb{P}^1$. The initial goal for this group was to investigate whether it is plausible that $\mathcal{F}_a$ and $\mathcal{F}_b$ have only finitely many constrained points in common when $a \neq b$. This generalizes a result for the quadratic unicritical map $z \mapsto z^2 + c$. Rather quickly, this group determined that a number of classical facts about single-map dynamics could not be taken for granted. For example, the standard proof that a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ has infinitely many preperiodic points does not immediately carry over, so the group spent a few days trying to find a proof that generalizes to unicritical correspondences. Additional considerations involved seeking to understand whether certain stronger notions of constrained point have bounded height, and what are fruitful analogues of multipliers and multiplicities of periodic points.

Height Computations.

A technique for the efficient computation of canonical heights for polarized morphisms was worked out by Call and Silverman, using a decomposition into local heights. This group met briefly on the final day to discuss the primary difficulty in extending this technique to canonical heights for correspondences: if a correspondence $C : X \rightrightarrows Y$ is defined over a number field k, then the orbit of a k-rational point under C does not typically remain in $Y(k)$.

This group sketched a number of examples and asked some questions about the nature of the canonical height that might make it possible to compute these heights efficiently, though this problem is still very much open.

K3-Surface Automorphisms.

This group studied the geometry of hypersurfaces X in $\mathbb{P}^{N+1}$ defined by the vanishing of a (generic) $(2, 2, \ldots, 2)$-form. The various projection maps $X \to \mathbb{P}^N$, which are double covers, yield $N + 1$ non-commuting involutions $\sigma_1, \ldots, \sigma_{N+1}$. For $N \geq 3$, some of the fibers of the projections are positive dimensional, leading to potential indeterminacy loci for the σ_i. Focusing on the case $N = 3$, they worked to prove that the σ_i extend to morphisms, and they started an analysis of the curve-counting function for the orbit of the $\mathbb{P}^1$ fiber. This included computing/analyzing the Néron–Severi group of the Calabi–Yau 3-fold X and the action of the σ_i on $\mathrm{NS}(X) \otimes \mathbb{Q}$.

Configurations.

Let X be a "configuration" of n points in a field k — i.e., a set of n distinct elements of k. Given any set-theoretic function $F : X \to X$, Lagrange interpolation tells us that there exists a unique degree-less-than-n polynomial $f(x) \in k[x]$ such that $f(a) = F(a)$ for all $a \in X$. For most choices of X, the degree of this polynomial will be exactly $n-1$. If $f(x)$ has degree less than $n - 1$, we call f a "low-degree endomorphism" of X. This group was interested in determining which configurations X have many low-degree endomorphisms or endomorphisms of exceptionally low degree?

If $X \subseteq \mathbb{C}$, then configurations with several exceptionally low-degree endomorphisms give us some insight into the conjectured uniform bounds on the total number of common prepriodic points shared between two dynamically distinct polynomials. If $X \subseteq \mathbb{Q}$, or more generally a number field, then a single low-degree endomorphism gives us insight into the conjectured uniform bound on the number of rational preperiodic points for polynomials.

Over the course of the week this group made progress in two directions:

- *Exotic embeddings of roots of unity.* If μ_n denotes the group of n-th roots of unity in $\mathbb{C}$, we say that an injective map $\ell : \mu_c \hookrightarrow \mu_{cd}$ is "exotic" if it is induced by a Möbius transformation such that $\ell(0) \neq 0, \infty$. If $d \geq 2$, define $\widehat{C}_d$ to be
$$\widehat{C}_d := \sup_{\substack{2 \leq \deg(f), \deg(g) \leq d \\ \mathcal{J}_f \neq \mathcal{J}_g}} |\mathrm{PrePer}(f, \mathbb{C}) \cap \mathrm{PrePer}(g, \mathbb{C})|$$
The existence of an exotic embedding $\mu_c \hookrightarrow \mu_{cd}$ gives a lower bound for $\widehat{C}_d$. The group discussed the problem of classifying all the exotic embeddings.
- *Exceptional Family of Six Point Configurations.* The group also analyzed the two parameter family of configurations given by $X_{a,b} := \{\pm a, \pm b, \pm(a-b)\}$ where $a, b \neq 0$ and $a \neq \pm b$. This family has 504 low-degree endomorphisms, independent of the values of a, b. They constructed a graph Γ that encodes the algebraic relationships between these low-degree endomorphisms of $X_{a,b}$. The graph Γ turns out to be highly symmetric and with few connected components, suggesting there is some interesting underlying structure which explains the exceptional properties of this configuration.