Nilpotent counting problems in arithmetic statistics

organized by
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Workshop Summary

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Arithmetic statistics has seen an explosion of progress in recent years. This workshop was structured to bring experts in several of these methods together to learn about them, and see where they might be combined to prove new results. The workshop had a particular focus on "nilpotent counting problems", meaning counting nilpotent extensions of global fields or finding the distribution of p-torsion of the class group of p^k-extensions, although discussion was not limited to these cases.

The working groups at the workshop fall roughly into three categories of problems:

(a) Number field counting, see , , ,
(b) Distribution of torsion in class groups of number fields, see
(c) Distribution of torsion in class groups over functions fields, see ,

This separation is merely thematic, as many of the problems worked on have aspects of the other categories. For example, the working group is focused on applying class group distribution results to count number fields, while working group is considering a problem over function fields for which the corresponding question over number fields has been solved.

Working Group Summaries

Here, we summarize several problems that were explored in the working groups.

Malle's conjecture for D_8 in degree 16.

The asymptotic number of D_8-extensions of the rationals is intimately related to the distribution of ${}_k[8]$ as k varies over quadratic fields. Gerth's heuristics for the distribution of $2_k[2^\infty]$ were recently proven in breakthrough work of Smith, and the purpose of this problem is to be a small test case to determine if and how Smith's methods can be applied to number field counting questions.

During the workshop, we spent a lot of time distilling the notation in Smith's paper (including ample consultation with Smith himself!). We settled on a simpler problem to address first: determining the asymptotic growth rate of

$$\sum_{|(k/)|\leq X} |{}_k[8]|$$

as k varies over quadratic fields. This is equivalent to counting D_8-extensions unramified over the quadratic subfield fixed by C_8. This asymptotic behaves like a mixed moment, involving both $|{}_k[2]| = 2^{\omega((k/))-1}$ and $2_k[8]$. The distributions of these factors are understood well individually (the former via contour integration and the latter via Smith's results).

We began work towards determining what, if anything, from Smith's results needs to be generalized to answer this new question.

During the course of the workshop, we worked down to a single result from Smith's work. This is a sieving result, showing that so-called "bad ideals" contribute a negligible proportion to counting functions. While at the workshop we determined that we require a slight strengthening of this sieve by restricting to the ideals with a bounded number of prime factors, which is the focus of our work after the workshop.

Lower Bounds for counting solvable extensions.

The goal of this group was to study lower bounds for solvable extensions in the number field situation. The easiest case of a solvable group that is not nilpotent is the dihedral group D_p of order $2p$ for a prime number p. In order to prove good upper bounds for the asymptotics it is necessary to know good upper bounds for the average p-torsion of class groups of quadratic number fields. For $p > 3$ this is a difficult problem. Nevertheless, it is known to prove the expected lower bounds in this situation, which was the motivation for this group.

In the beginning we studied meta-cyclic groups, and similar ideas as in the dihedral case can be applied to prove the expected lower bounds. We also looked at the alternating group A_4 which is a meta-abelian case. Here we worked out the lower bounds including log-factors.

In the more general situation of solvable groups we did not find a unifying approach. We studied some groups that need more than 2 steps.

Counting number fields ordered by ^{1} embedding.

Let $f = f_0x^n + f_1x^{n-1}y + \ldots + f_ny^n$ be a binary n-ic form with integral coefficients. Define the height of f to be $H(f) := \max(|f_i|)$. To f we can associate the number field $K_f := \mathbb{Q}[x]/f(x, 1)$. During the workshop we considered the following questions:

(1) When do two binary n-ic forms define isomorphic number fields?
(2) Define the $\mathbb{P}^1$-height of a number field K to be the minimum height $H(f)$, where f ranges over binary n-ic forms with $K = K_f$. How many degree-n number fields are there with $\mathbb{P}^1$-height bounded by H?

We have made some progress for $n = 2, 3$, and 4 using known parametrizations of binary rings. We also explored a representation theoretic construction that recovers the known parametrizations. This allowed us to make some progress towards solving questions (1) and (2) for general values of n.

A new case of Shankar–Tsimerman's counting heuristic.

We set up a framework for generalizing the Shankar–Tsimerman heuristic for counting S_n number fields ordered by discriminant, to the case of fields with Galois group $C_2 \wr S_n$. Here the monogenic number fields that we consider are those generated by an element whose minimal polynomial is even (of the form $x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0$). We discussed what invariant we should order our fields by, analogously to the discriminant in the Shankar–Tsimerman heuristics, and came up with a candidate defined as the covolume of a certain lattice. We observed that in the case of $D_4 = C_2 \wr S_2$ extensions this invariant is the conductor (up to a power of two), so we can test our heuristic by comparing it against the Altug–Shankar–Varma–Wilson results on counting D_4 fields by conductor. We formulated an analogue of Shankar-Tsimerman's main heuristic assumption, and investigated the p-adic integrals appearing in the calculation of the local densities. We hope that as in Shankar–Tsimerman, these integrals would simplify nicely, and made progress towards confirming this.

Liu's Conjecture for the $\Gamma = C_2^2$ component of $_k[2^{\infty}$
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Our general goal is to understand the distribution of p-part of class group of Γ-extensions of $\mathbb{Q}$, when Γ is a finite abelian p-group. When $\Gamma = \mathbb{Z}/p\mathbb{Z}$, this question is related to Gerth's conjecture, which is proved by Smith and Koymans–Pagano. For general Γ, Liu's recent work proposes a new way to look at these class groups in terms of modules over DVRs, and proposes conjectures for the distribution. During the workshop, we studied the most simple case: $\Gamma = \mathbb{Z}/2\mathbb{Z} \times /2\mathbb{Z}$; in particular, we spent lots of time understanding Liu's conjecture in this case and comparing with the framework of Smith's method. We realized that there is not a natural comparison between these two approaches, as they classify the module structure of class groups in completely different ways. Shortly after the workshop, we figured out that Liu's conjecture in the case that Γ is elementary abelian can be reduced to the $\mathbb{Z}/p\mathbb{Z}$ case, so using Smith's work on Gerth conjecture, we should be able to prove that case. For general Γ, we hope to show that Liu's conjecture can be reduced to cyclic cases, and also generalize Smith's method to cyclic cases.

Distribution of the 4-rank of $_k$ over quadratic function fields.

In this project group, we focused on formulating a geometric approach to understanding distribution of 4-ranks of class groups of quadratic extensions over global function fields. We explored the problem with the following key directions.

- Understand geometric properties of the moduli spaces of 2 and 4-torsions of Jacobians of hyperelliptic curves with n Weierstrass points, such as their numbers of geometrically connected components.
- Formulate an adequate analogue that computes the distribution of 4-ranks of class groups of quadratic extensions over number fields.
- Obtain numerical data to understand the order of error terms for computing the desired distributions.

The project group was able to make some progress in all three directions. Some of the key ingredients the project group used involve the Burau representation of the braid group on n strands, the Grothendieck-Lefschetz trace formula, and parametrization of D_4-extensions paired with bi-quadratic sub-extensions over global fields.

Distribution of $_k[3]$ for k varying over S_3-cubic function fields.

Our general problem is to understand distributions for $\mathrm{Cl}\,K[3]$ for S_3 cubic extensions $K/\mathbb{F}_q(t)$. Our problem thus concerns occurrences of unramified $\mathbb{Z}/3\mathbb{Z}$-extensions L/K, which in turn can be translated into counting $\mathbb{F}_q$-points on Hurwitz schemes Hur_T parameterizing Galois covers of $\mathbb{P}^1_{\mathbb{F}_q}$ with automorphism group $T = \{(a, b, c, \sigma) \in \mathbb{Z}/3\mathbb{Z} \wr S_3 = \mathbb{Z}/3\mathbb{Z}^{\oplus 3} \rtimes S_3 : a+b+c = 0\}$. During the week of the workshop, we identified the four "admissible" conjugacy classes of T — the conjugacy classes which can occur as monodromy types of the branching of the tower extension $L/K/\mathbb{F}_q(t)$ (or more accurately, of the branching of the corresponding covers of curves). Doing so was the first step in understanding the number of components, which contribute dominating parts to the point counts, of the relevant Hurwitz schemes. In particular, the identification of admissible conjugacy classes allowed us to observe a phenomenon — that the number of connected components of the Hurwtiz scheme of T-covers grows exponentially in relation to the number of connected components of the Hurwitz scheme of S_3-covers — analogous to one from Yuan Liu's work presented during the workshop. However, this is merely an asymptotic analysis, counting clopen parts $\mathrm{Hur}_T^{m,n_1,n_2,n_3}$ in Hur_T, where m, n_1, n_2, n_3 refer to the number of branch points whose monodromy types are of each of the four admissible conjugacy classes of T. To further granulate this analysis by counting the number of components of $\mathrm{Hur}_T^{m,n_1,n_2,n_3}$, we studied the second group homology $H_2(T, \mathbb{Z})$ with the hope of applying Melanie Matchett Wood's work *An algebraic lifting invariant of Ellenberg, Venkatesh, Westerland* to bound the number of components.

Future Plans

Many of the groups described above are continuing the research they started at AIM. We expect multiple papers to eventually be submitted based on this work, though the work that was started at the workshop is all still in progress. One project that was started earlier was completed at the conference, and the preprint is now available (Koymans and Pagano's paper "Hilbert's tenth problem via additive combinatorics").

We created a Zulip to be a community space for all of the working groups and other researchers in arithmetic statistics. This has threads for each of the working groups and also for other discussions in the area. We think this has the potential to help create new opportunities for collaborations to form, even though largely many collaborations still rely on email for communication.