Noncommutative inequalities

organized by
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Workshop Summary

Noncommutative (nc) inequalities in the sense of positive definite matrices appear in a number of currently vibrant areas of mathematics and its applications. Some of these include free analysis and control systems, random matrix theory, mathematical physics, computational complexity and the growing field of quantum information theory. The corresponding disparate communities have each made recent major contributions to the theory of nc inequalities. The goal of this workshop was to bring together dynamic practitioners in these areas to identify problems of mutual interest and develop collaborative research infrastructure.

The workshop was held the week of June 14 through 18, 2021. Each morning there were two talks with the aim of introducing participants to the ideas, goals and important problems in the various subjects. After a problem session on Monday afternoon, during the remaining afternoons the participants worked on some of these problems in four research groups. The speakers were

(1) Monday. Bill Helton and William Slofstra.
(2) Tuesday. Jurij Volčič and James Pascoe.
(3) Wednesday. Nikhil Srivastava and Visu Makam.
(4) Thursday. Rafael Oliveira and Eli Shamovich.
(5) Friday. Anna Vershynina and Anand Natarajan.

Helton, Pascoe and Shamovich work in various aspects of noncommutative analysis. Volčič's interests straddle noncommutative analysis and algebra, with some expertise in invariant theory. Makam and Oliveira presented connections between invariant theory, computational complexity and noncommutative algebra. Slofstra and Natarajan spoke on aspects of quantum information theory, Natarajan from the physics perspective and Slofstra from the mathematical point of view. Srivastava has expertise related to random matrix theory. Vershynina introduced participants to the issue of noncommutative convexity for trace functions that arise in quantum information theory and related data processing inequalities.

A meeting wide discussion on Monday afternoon generated the following problems of potential mutual interest.

(1) Is there a positive element in the group algebra of the cartesian product of the free group with itself that is not a sum of squares?
(2) Find tractable algebraic certificates for trace positive noncommutative polynomials.
(3) Determine degree bounds and complexity aspects of noncommutative Positiv- and Nullstellensätze.
(4) Develop algorithms for noncommutative rational identity testing with efficiency estimates based upon algebraic circuits.

(5) Determine those pure trace polynomials that have a convex positivity set.
(6) Investigate the distribution of hyperbolic eigenvalues in a Gaussian random direction for hyperbolic polynomials.
(7) Is there a noncommutative analog of hyperbolic polynomials?
(8) Connect unique solvability and sums of squares representations for quantum games.
(9) An algebra, versus ideal, containment version of a Nullstellensatz for noncommutative polynomials.
(10) Is it true that the algebra of a 2-XOR game is finite dimensional if and only if the game is uniquely solvable?
(11) Detect composition of noncommutative rational functions through rationality and simplicity of their eigenvalues.
(12) Determine the convexity properties of a particular family of functions of matrix variables related to the data processing inequality from quantum information theory.
(13) Do ranks of evaluations of linear matrix pencils separate matrix tuples up to similarity?
(14) Determine canonical realizations for noncommutative matrix monotone functions.

Problems (1), (3), (8) and (10) concern quantum information theory, quantum games and quantum complexity; Problems (2), (3), (5), (7), (9), (11) and (14) are connected to noncommutative analysis and noncommutative real algebraic geometry; Problems (4), (9) and (13) involve invariant theory; Problem (6) comes out of random matrix theory; Problems (3) and (4) pertain to computational complexity; and finally, Problem (12) refers to a data processing inequality from quantum information theory.

In the afternoons, Tuesday through Friday, the participants divided into four research groups identified as XOR, Linear Pencils, Approximation and Trace Convexity for organizational purposes. There was some movement of participants between groups during the week.

The XOR group studied synchronous two player quantum games. Connected to Problems (1), (8) and (11), they considered games that have a perfect quantum solution, but not classical solution, and tried to prove the conjecture that there exists a perfect solution whose state L is tracial, that is, $L(AB) = L(BA)$, by applying the convex Positivstellensatz of Helton-Klep-McCullough to this problem. The main difficulty is the partial commutativity between Alice and Bobs operators. However, there is a lift to a close/equivalent noncommutative problem, to which the convex Positivstellensatz does apply. This lift gives an algebraic certificate with tightly bounded complexity. The XOR group worked with these bounds, but they did not imply the desired results. The fact that the bounds obtained are sharp in various senses suggests that the approach is unsound or possibly that the conjecture is false. Work on this problem continued for about a month beyond the workshop.

The Linear Pencils group focused on Problem (14). This question is a two-sided version of the Hadwin-Larson conjecture of 2003, and arises from operator theory. Harm Derksen noticed the connection between this problem and a topic in the representation theory of associative algebras, which led to the affirmative answer to this question. Furthermore, there is a bilinear bound mn on the size of linear pencils whose ranks separate similarity orbits of $n \times n$ matrix m-tuples. The synergistic nature of the workshop was therefore especially beneficial in this case. Next, the group considered the general version of the Hadwin-Larson conjecture, which relates rank inequalities and orbit closure membership. The group established that it is false. The counterexample again came from representation theory, this time from the study of module degenerations. The group then considered a relaxation of the Hadwin-Larson conjecture (rank inequalities vs orbit closure membership for ampliations), but its veracity stayed unresolved. The work started in this group continued over the summer, and resulted in the paper *Ranks of linear matrix pencils separate simultaneous similarity orbits* by H. Derksen, I. Klep, V. Makam and J. Volčič, arXiv: 2109.09418.

The Approximation group considered versions of Problem (6). In particular, they tried to discover a derandomized version of how small the eigenvalue condition number is near a badly conditioned matrix as opposed to recent randomized results in these directions. The problem has connections to numerical inversion.

On Monday the Trace Convexity group determined that Problem (5) and several variants were not viable. The remainder of the week, this group delved into Problem (12) and developed several promising conjectures bolstered by numerical evidence. Since the workshop, these conjectures, and further results, have been proved and a paper is near completion.

A number of aspects of Problem (9) were settled on Monday. Suppose $d \geq 1$, $n \geq 1$, $x_1, \ldots, x_d$ are freely noncommuting indeterminates and $p = (p_1, \ldots, p_n)$ is a tuple of free polynomials in terms of $x_1, \ldots, x_d$. The problem was to investigate the conjecture: *Suppose q is a free polynomial. If, for any tuples of square matrices (of the same sizes) X and Y, the equality $p(X) = p(Y)$ implies $q(X) = q(Y)$, then q in the unital algebra generated by $p_1, \ldots, p_n$.* The conjecture is true when $n = d$. However, a counterexample was provided for $n = 4$ and $d = 2$: $p = (x, xy, yx, y + yxy)$ is injective but y is not in the unital algebra generated by p.