Stochastic methods for non-equilibrium dynamical systems

organized by
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Workshop Summary

1. Overview

The focus of this workshop was the study of the statistical properties of dynamical systems, in particular systems of physical interest that can shed light on problems in non-equilibrium statistical mechanics. Among the topics explored were various types of deterministic and random billiard systems, non-uniformly hyperbolic systems, transport properties in classical systems, and stochastic thermodynamics.

The event brought together 26 participants including senior experts and many young researchers from the US, France, Canada, Australia, UK, and Italy, with research interests ranging from ergodic theory, probability and dynamical systems to very applied stochastic systems in physics, chemistry, biology, and ocean and atmospheric science.

A number of the goals of this workshop were realized: it brought to the attention of this community of researchers some recent theoretical advances and new tools of analysis for the study of non-uniformly hyperbolic (chaotic) dynamical systems; it fostered a discussion of what constitutes non-equilibrium phenomena in dynamics and statistical physics, and what are the important problems in this field from the different perspectives of, on the one hand, Hamiltonian (deterministic) mechanical systems, particularly billiard systems, and on the other chemical and biological systems; it proposed new types of mathematical models that may help bring these different perspectives closer together and advance our understanding of the subject; it formed new collaborations that will advance our understanding of these important topics in the near future.

2. Summary of talks

There were two morning talks each day, many of which were introductory lectures on a given topic. Here is a brief outline.

- Hong Qian's presentation concerned *Nonequilibrium Processes: Deterministic and stochastic dynamics, and emergent thermodynamics.* Here's an overview: The nature of nonequilibrium system is to have sustained net transport of mass, momentum, charge, and/or energy. To understand non-equilibrium systems, therefore, one needs to first be able to formulate a mathematical theory of conservation law of these quantities in equilibrium. In a stochastic mathematical description of dynamics, Qian introduced the notion of entropy and entropy production in terms of the Kullback-Leibler divergence. While entropy characterizes the "randomness in the dynamics per step," the entropy production characterizes the reversible, or irreversible nature of dynamics with respect to an proper time reversal. These two concepts match the

theory of *transport coefficient* and the theory of *transport process with non-zero net flux(es)*, respectively. Onsager's physical insight was that entropy production equals flux times the force that is responsible for the flux.

- Sandro Vaienti spoke on the subject of *Extreme Value Theory* and its renewed interest in describing statistical properties of dynamical systems. He presented some recent results in the framework of random and non-autonomous (sequential) dynamical systems. The novelty of his contribution is in the lack of stationarity of the random processes for which he computes the distribution of the maximum. The random situation is described by taking as a probability the conditional measure on the fibers of the associated skew system; the sequential scenario is given by maps which verify a certain number of assumptions as described by the Conze-Raugi theory.
- Françoise Pène considered the planar periodic Lorentz process describing the motion of a point particle moving in the plane and bouncing on convex obstacles (with a periodic configuration of obstacles) under the assumption of a finite horizon, which means that the particle cannot go to infinity without hitting an obstacle. Using Young towers, she establishes precise local limit theorems in this context and presents two applications of them. The first is a study of the number of different obstacles visited by the Lorentz process up to time t for which she proved an almost sure convergence with a suitable normalization. The second application is the asymptotic behavior of the Lorentz process in random scenery. Here she associated to each obstacle a random variable. These random variables are assumed to be independent identically distributed and independent of the Lorentz process. Assuming that the particle wins the random variable associated to an obstacle each time it bounces on it, the Lorentz process in random scenery describes the evolution in time of the total amount won by the particle.
- Renato Feres spoke on *New billiard models in stochastic thermodynamics*. He described results obtained in collaboration with Tim Chumley, Scott Cook, and Hong-Kun Zhang. Among the topics covered he showed how a certain class of Markov chains derived from billiard systems can illustrate thermodynamical behavior such as steady-state generation of work and entropy. He also gave examples of so-called Knudsen diffusion processes for random billiard systems for which one can obtain exact values of the diffusion constant.
- Leonid Bunimovich described some connections between central problems in statistical mechanics and the study of mathematical billiards. For example, billiards are a natural physical model from which to derive irreversible hydrodynamic or kinetic equations from reversible microscopic dynamics. Bunimovich also described connections between billiards, transport phenomena and conservation laws for mass (a minimum of 1 billiard particle required), momentum (a minimum of 2 billiard particles required), and energy (a minimum of 3 billiard particles required). Some progress on these problems was summarized, but many fundamental questions remain open. Finally, Bunimovich gave an overview of hyperbolic billiards and described new mechanisms of hyperbolicity in billiard and particle systems.
- Viviane Baladi spoke on *Exponential decay of correlations for Sinai billiard flows*. She presented joint work with M. Demers and C. Liverani in which they proved exponential decay of correlations for two-dimensional Sinai billiard flows with finite horizon.This was achieved by studying the spectrum (resonances) of a transfer

operator on a suitable Banach space of anisotropic distributions, combining techniques of Dolgopyat and Liverani (as adapted in a previous work of Baladi-Liverani on piecewise hyperbolic contact flows) and methods of Demers-Liverani (as adapted in a previous work of Demers and Zhang giving a new proof of Young's result of exponential decay of correlations for Sinai billiard maps).

- Alex Grigo presented work on dynamical systems with Euclidean symmetry, focusing on the behavior of spiral wave solutions in certain infinite dimensional systems. Here is his overview: one of the central questions in statistical mechanics is to derive macroscopic equations for (interacting) many particle systems from their microscopic evolution equations. In particular, deriving expressions that allow to compute transport coefficients in terms of parameters that enter the microscopic description are of great interest. Mathematical models of physical relevance where such questions can be answered in a mathematically rigorous way are difficult to come by. In my talk I presented a billiard model of a granular gas/media for which I am able to derive the macroscopic equation for the evolution of the total energy (which is not preserved in this model). Although the mathematical proof of validity I can give only for 2 particles, the argument should be provable also for any number of particles, but this is at present out of reach. However, taking this for granted I can show that the corresponding result agrees with the prediction of the corresponding Boltzmann equation in the Boltzmann-Grad limit. Certainly there are many technical challenges left to resolve, which will be part of future research.
- Jacopo de Simoi spoke on *Decay of correlations in fast-slow partially hyperbolic systems*. He showed the existence of an open class of partially hyperbolic smooth local diffeomorphisms of the two-torus which admit an unique SRB measure satisfying exponential decay of correlations and obtain nearly optimal estimates for the rate of decay of correlations. He focused on the more dynamical aspects of the proof and discussed some related questions which still remain open. His work is part of a joint project with Carlangelo Liverani.
- Ian Melbourne gave an overview of results on superpolynomial decay and polynomial decay for nonuniformly hyperbolic circle extensions, emphasizing the similarities and differences between the situations for flows and circle extensions. He considered large classes of examples, including Anosov and Axiom A dynamical systems, billiards, intermittent maps, (geometric) Lorenz attractors, and Hénon-like attractors.
- Gary Froyland spoke on a program of work to establish existence and stability results for linear cocycles in the semi-invertible situation—where the driving mechanism is invertible, but the linear actions may be non-injective—and to create numerical methods to apply to real-world models and data. The "existence of Oseledets splitting" results provide a stronger multiplicative ergodic theorem than the "classical" theorems, which only guarantee the existence of measurable Oseledets filtrations. The stability results concern continuity properties of the Lyapunov exponents and their corresponding splitting elements when the linear actions are subjected to a variety of perturbations. The applied motivations for this work are the detection and tracking of so-called coherent structures in time-dependent dynamical systems, and Gary also reported on the application of these constructions to fluid flow in the ocean and atmosphere. This is joint work with Cecilia González Tokman, Christian

Horenkamp, Simon Lloyd, Adam Monahan, Anthony Quas, Vincent Rossi, Naratip Santitissadeekorn, Alex Sen Gupta, and Erik van Sebille.

3. Summary of working groups

In the afternoon activities, participants were divided into approximately 6 working groups whose composition changed from day-to-day. The following is a brief description of some of the topics explored during the afternoon discussions.

- Problem 1: One working group chose to study problems connected with the determination of *transport coefficients* for processes defined in terms of random billiard systems. This general problem is connected with topics such as *fluctuation-dissipation* theorems, *Green-Kubo formula*, and *Einstein relation* in non-equilibrium statistical physics. A concrete set-up investigated by the group was as follows: one considers the random flight of a charged billiard particle in a straight cylindrical channel subject to a constant electric field. The interaction between particle and channel surface is defined by a dynamic/geometric model that naturally gives rise to thermostatic behavior (thus fixing the temperature of the channel wall.) From this microscopic interaction model one obtains a random surface scattering operator and a random billiard system. The problem then is to prove that this system satisfies Ohm's law and derive the value of conductance as a function of structural parameters of the microscopic interaction, such as mass-ratios, curvatures, etc. The group made some progress in understanding the key issues involved and setting up a reasonably tractable but representative model system.
- Problem 3: Another working group investigated whether, in general, in the setting of random dynamical systems theory; are the normalizing constants in annealed and quenched central limit theorems (CLT) the same? To give some insight into this problem suppose we have a set of maps $\{T_1, ..., T_n\}$ which act on a space X and we choose them independently according to a probability distribution vector, $(p_1, ..., p_n)$ where $\sum_{i=1}^n p_i = 1$ and let the resulting composition of maps act on X. This induces a measure ν on the shift space $\Omega = \{1, ..., n\}^N$. Points $\omega = (.\omega_1\omega_2...\omega_n...)$, $\omega_i \in \{1, ..., n\}$ represent a sequence T_{ω_1}, T_{ω_2} ,..., of maps. Under certain assumptions on the maps it can be shown that there is a stationary natural invariant measure μ induced on X by this random dynamical system (RDS). Given an observable on X, $\phi : X \to R$ we may consider whether the RDS exhibits a CLT with respect to the invariant measure $\nu \times \mu$ (an annealed CLT), namely $(\nu \times \mu)\{(\omega, x) \in \Omega \times X : \sum_{j=1}^n \frac{1}{\sigma_j}[\phi(T_{\omega_n} \dots T_{\omega_1} - a_j] \in (a, b)\} \to \frac{1}{sqrt2\pi} \int_a^b e^{-t^2}2$. Or a quenched CLT may hold and for ν a.e. sequence of maps $T_{\omega_n}...T_{\omega_1}$, $\mu\{x \in X : \sum_{j=1}^n \frac{1}{\sigma_j(\omega)}[\phi(T_{\omega_n} \dots T_{\omega_1} - a_j(\omega)] \in (a, b)\} \to \frac{1}{sqrt2\pi} \int_a^b e^{-t^2}2$. If the observable satisfies $\int \phi d\mu = 0$ then in the annealed CLT we may take $a_j = 0$ for all j. The question investigated was under which settings is $a_j(\omega) = a_j$ and $\sigma_j(\omega) = \sigma_j$ for ν a.e. $\omega \in \Omega$? There had been recent examples in the literature in which annealed and quenched CLTs had been proved for maps T_i which preserve the same invariant measure μ and in this case the scaling constants were almost surely constant. The group considered simple examples of RDS consisting of β transformations (which have different invariant measures). It was realized that a recent PhD thesis by Romain Aimino gave, in some sense, an

example in this setting where in the quenched CLT the scaling constants cannot be chosen almost surely constant. There are many open questions however remaining.

- Problem 4: The application of martingale theory to the study of statistical properties of dynamical systems was pioneered by M. Gordin in 1969. The classical approach is to reduce an invertible hyperbolic system to a non-invertible one by 'quotienting' out the stable direction and then study the action of the transfer operator (the dual to the action of composing an observable with the transformation). The martingale approximation using the Gordin approach expresses an observable on the system as the sum of a reverse martingale difference and a coboundary on this reduced space. Recent work has investigated the action of the transfer operator directly on certain anisotropic Banach spaces and for many reasons this approach is preferable (in particular it allows perturbation). However the martingale technique of Gordin has not yet been adapted to this context. This discussion group discussed ways of modifying Gordin's approach to this setting. Some ideas were suggested but no definitive answer emerged; however, several participants in the workshop expressed an interest in continuing to work on this problem and they have plans to meet in 2016 to make progress on this research question.
- Problem 5: This group discussed two related problems: A quenched Central Limit Theorem for a randomly perturbed billiard system, and a local limit theorem for both deterministic and perturbed billiards. Françoise Pène began the discussion by making the connection between the two limit theorems and outlining a general plan which, if successfully implemented, should yield the desired limit theorems.

  The general strategy relies on getting uniform estimates for a family of twisted transfer operators. The discussion then turned to the Banach spaces recently constructed by Hong-kun Zhang and Mark Demers for which the associated transfer operators for some classes of dispersing billiards have been shown to enjoy a spectral gap. Zhang and Demers both described their results on perturbations and the control they had proved on some similarly twisted transfer operators. The known results are for the annealed (or averaged) random dynamics, and discussion proceeded to identify the elements which would have to be strengthened in order to prove the analogous quenched version and the local limit theorem.

  Some initial estimates were worked out by the group during the second day of discussions and a draft is circulating among several members of the discussion group. This group has initiated a collaboration that we expect will yield a paper proving the above-mentioned limit theorems in the near future.
- Problem 6: Since the question was very broadly phrased, the group first agreed to define 'slowly mixing' to refer to any behavior occurring at a subexponential rate. They discussed the construction of systems which mix at stretched exponential rates, of which there are few examples compared to systems which mix at exponential or polynomial rates. Some ideas were exchanged regarding various types of limit laws that might depend sensitively on the rate of mixing.

  M. Todd described a current project with another workshop participant, M. Demers, which is exploring connections between open systems, i.e. systems in which mass is allowed to escape, and return times statistics, which quantifies rates of return to small sets. When studying the return times to a set, one typically takes a 'diagonal' limit, scaling time as inversely proportional to the measure of the set in

question to obtain a limit law; by contrast, the open system approach fixes a set, then first takes a limit in time to obtain an escape rate, followed by a derivative of the escape rate as the hole shrinks to a point. In some cases, the same limit law is obtained via both procedures. This can be generalized to limits along other paths in parameter space and the current project seeks to differentiate among such paths according to the (subexponential) rate of mixing. N. Hayden showed interest in this project and suggested generalizing the approach to ϕ and ψ-mixing systems in a symbolic setting.

- Problem 7: The group comprised G. Froyland, C. Gonzalez-Tokman, N. Hayden, J. de Simoi, and S. Vaienti. The group discussed dynamical systems in nonautonomous or random settings (ergodically forced dynamical systems). In particular, the group considered a functional analytical approach to investigate large deviations and limit theorems in such settings. Several important questions were formulated precisely and the group members' combined expertise and particular combination of skills in large deviations, limit theorems and multiplicative ergodic theory seems likely to resolve some of them, although there are no concrete results as of yet. The group has remained in contact by email.
- Problem 9: The group comprised V. Baladi, M. Demers, F. Pène and M. Todd. The group began by performing a literature search to determine what was known regarding topological and metric entropy for dispersing billiards. We found that for a finite horizon Lorentz gas, both map and flow have finite topological entropy, while for an infinite horizon Lorentz gas, the flow has finite entropy while the map has infinite entropy.

  The discussion then centered on the billiard map corresponding to a finite horizon Lorentz gas, and the possibility of obtaining the measure of maximal entropy via transfer operator techniques. Such techniques have been used successfully to obtain measures of maximal entropy for expanding maps and for Anosov diffeomorphisms. Now that the transfer operators associated with dispersing billiards can be directly analyzed on appropriate Banach spaces of distributions, the goal of the group was to determine if these techniques would generalize to dispersing billiards.

  The group identified at least two nontrivial difficulties with implementing this approach: (1) due to the fact that the stable and unstable foliations for billiards are only measurable, the required potential for the modified transfer operator might be no better than a measurable function - this could pose severe problems for the implementation of the constructed norms; (2) the blow-up of the derivative near tangential collisions typically requires the introduction of a countable family of homogeneity strips accumulating on the singularities in order to regain bounded distortion properties of the dynamics - unfortunately, with the modified potential, certain required dynamical sums over these homogeneity strips may diverge.

  The group discussed some possible work-arounds to these obstructions, but it soon became apparent that this was a long-term project that would not be solved quickly. Two group members have a research visit planned in 2016 to begin work on this problem.