Nonlinear PDEs in real and complex geometry

organized by
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Workshop Summary

The aim of this workshop was to bring together experts in nonlinear PDEs motivated by problems in real and complex geometry. While major progresses have been made recently, especially in the study of canonical metrics in Kahler geometry, many new and exciting directions are also opening up. The workshop participants had varied backgrounds, and included experts in complex geometric, Riemannian geometric and "pure" PDE, with the idea of bringing new perspectives to problems in nonlinear PDE.

The workshop followed the usual AIM format. There were two talks each morning. We held an open problem session on the Monday afternoon, generating around 25 problems on wide-ranging topics including Monge-Ampère equations, complex geometry and geometric flows. In all cases the problems centered on a nonlinear PDE, and most were connected to or motivated by geometry. In the afternoons of Tuesday to Friday we broke up into groups of 2-6 people to discuss some of these problems. The groups were fluid, with people moving and creating new groups from one day to the next. Below are brief reports on the problems that were discussed.

(1) **The Kähler-Ricci flow with finite existence time.** The original problem that was discussed was whether the Kähler-Ricci flow should always have bounded diameter if a singularity occurs in finite time. Since this appeared to be a somewhat intractable problem, the members of this group moved onto a related question of whether finite time extinction (meaning that the diameter of the manifold tends to zero at finite time) implies that the underlying complex manifold is Fano and the initial class a positive multiple of the anticanonical class. This question has been answered by J. Song in the case of rational classes on algebraic manifolds, but remains open in general.
(2) **The Calabi flow on toric surfaces.** It still remains an open question as to whether the Calabi flow (a fourth order parabolic PDE on Kähler manifolds) exists for all time, even in the case of toric surfaces. The members of this group were optimistic about making some progress under certain symmetry conditions, which could help elucidate the key difficulties.
(3) **Donaldson's modified geodesic equation**. In a 1999 paper, Donaldson proposed the study of an equation $\ddot{\varphi} - \frac{1}{2}|\nabla\dot{\varphi}|^2 = \lambda R_\varphi$ on $M \times [0,1]$ for M a compact Kähler manifold, where λ is a (presumably negative) small constant λ. This is a modification of the well-known geodesic equation on the space of Kähler metrics. It was noted that this equation (which is fourth order) can be written as a coupled system of parabolic second order equations. From this, some preliminary *a priori* estimates follow. However, basic questions remained, including the well-posedness for appropriate boundary data, and further study is needed.

(4) **The equation** $\det(D^2u)u^{ij}u_iu_j = 1$. It was quickly discovered that this equation, after a suitable change of coordinates, is equivalent to the Guass curvature flow equation which has already been extensively studied.

(5) **Solutions of the complex Monge-Ampère equation with quadratic bounds.** If a plurisubharmonic function u on $\mathbf{C}^n$ solves the complex Monge-Ampère equation $\det(u_{i\bar{j}}) = 1$ and satisfies the bounds $c(1+|z|^2) \leq u \leq C(1+|z|^2)$ for positive constants c, C, is it necessarily quadratic? A suggestion was made to consider the tensor $u_{i\bar{j}}$ as a metric on $\mathbf{C}^n$ and employ compactness results for such manifolds, but this approach quickly led to difficulties due to the possible non-completeness of the metric. A second suggestion was to modify the problem by considering instead the (harder) fourth order equation $R =$ constant and replacing the bounds $c(1 + |z|^2) \leq u \leq C(1 + |z|^2)$ by suitable bounds on the metric tensor $u_{i\bar{j}}$. Several members of this group were keen to follow this up with further study.

(6) **Behavior of flows of Hermitian metrics on the Hopf surface.** As the simplest Class VII surface, the Hopf surface $(\mathbf{C}^2 - \{0\})/(z \sim 2z)$ provides a good model to study flows of Hermitian metrics on non-Kähler manifolds. An outcome of this discussion was that under T^3 symmetry conditions the behavior of the Chern-Ricci flow converges to the same limit as in the evolution of the standard Hopf metric, and some members of this group were optimistic about making further progress.

(7) **The Goldberg conjecture.** This long-standing conjecture states that an almost-Kähler Einstein metric on a compact manifold is Kähler. This group considered a "local" approach to this conjecture by investigating deformations of Kähler-Einstein metrics with negative scalar curvature. A result of Koiso states that when the complex deformation space is unobstructed, then any nearby Einstein metric is actually Kähler. The group plans to study the obstructed case further, with the hope of either proving a local version of Goldberg's conjecture, or finding a counterexample.

(8) **The Hamiltonian Stationary Lagrangian equation**. Given a smooth function $u : \Omega \to \mathbf{R}$ on a domain $\Omega \subset \mathbf{R}^n$, the graph of the gradient Du can be viewed as a Lagrangian submanifold in $\mathbf{C}^n$. The Hamiltonian stationary Lagrangian equation is the Euler-Lagrange equation for the functional given by the area of this submanifold. It is a fourth order elliptic equation for u. The group considered the problem of finding suitable boundary conditions for u when Ω is the unit ball, in order to ensure the existence of a unique solution, and are planning to keep pursuing this question.

Overall the organizers feel that the workshop was a success. It brought together mathematicians working in different fields, exposing them to new ideas and forging new collaborations. We are optimistic that workshop participants will continue to work on these questions, and that the list of problems, which is available on AIM's website, will be a valuable resource for the field.

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