LIST OF OPEN PROBLEMS
AIM WORKSHOP: MATHEMATICAL ASPECTS OF PHYSICS WITH NON-SELF-ADJOINT OPERATORS

***Open problems suggested during the meeting**

1. Yaniv Almog

1.1. **Completeness of eigenfunctions for Schrödinger operators with complex potentials.** For $\alpha > 0$ consider $A_\alpha := -\mathrm{d}^2/\mathrm{d}x^2 + \mathrm{i}|x|^\alpha$ in $\mathbb{R}$ (or $\mathbb{R}_+$ with Dirichlet boundary condition at 0). If $\alpha > 2/3$, then it is known that the eigenfunctions form a complete system.
Open problem: Is the same true for $0 < \alpha \leq 2/3$?

1.2. **Magnetic Schrödinger operator.** Consider

$$A := -\frac{\partial^2}{\partial x^2} - \left(\frac{\partial}{\partial y} - \frac{\mathrm{i}x^2}{2}\right)^2 + \mathrm{i}cy, \quad \mathcal{D}(A) := H_0^1(\mathbb{R}_+^2) \cap \{u : Au \in L^2(\mathbb{R}_+^2)\}$$

where $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.
Open problem: Is $\sigma(A) \neq \emptyset$?
It is known that $\sigma(A) \neq \emptyset$ if $|c| << 1$ or $|c| >> 1$.

2. Lyonell Boulton

2.1. **Schauder bases of periodic functions and multipliers.** Let $e_n(x) := \sqrt{2}\sin(n\pi x)$. Then $\{e_n\}$ is a Schauder basis of $L^p(0, 1)$ for all $p > 1$. Let $f \in C(\mathbb{R}, \mathbb{C})$ satisfy $f(x + 2) = f(x)$, $f(-x) = -f(x)$, $f(1/2 + x) = f(1/2 - x)$ and define $f_n(x) := f(nx)$. Let $A : L^p(0, 1) \to L^p(0, 1)$ be the linear extension of the map $Ae_n = f_n$. Then $\{f_n\}$ is a Schauder basis of $L^p(0, 1)$ if and only if $A : L^p(0, 1) \longrightarrow L^p(0, 1)$ is a bounded operator with a bounded inverse. Let $\{c_k\}$ be the Fourier coefficients of f. Then A can be written as $A = \sum_k c_k M_k$ where M_k are the linear extensions of the map $M_k e_n = e_{kn}$.
Open problem: Find necessary and sufficient conditions on $\{c_k\}$ for $0 \notin \sigma(A)$ whenever $p \neq 2$.

3. Amin Boumenir

3.1. **Non-self-adjoint inverse problems.** We are interested in identifying a non-self-adjoint operator associated with an evolution equation (parabolic or hyperbolic) through "observations" of the solution as time evolves. Thus for example in a certain Hilbert space we have

$$u'(t) = Au(t) \quad \text{and } u(0) = f \tag{1}$$

where, for simplicity, we assume that

$$A = L + B$$

with L is a given (known) self-adjoint operator with "nice properties" while B is an unknown non-self-adjoint perturbation. For example $Ay(x) = y''(x) - q(x)y(x)$ or $Au = \Delta u - q(x)u$ with $\operatorname{Im} q(x) \neq 0$. We assume that we can observe the solution through a functional $\langle \cdot, g\rangle$ say

$$\omega(t) = \langle u(t), g\rangle.$$

For example if $u(x, t)$ is the solution of a heat equation, where $x \in \Omega \subset \mathbb{R}^n$, and $p \in \partial\Omega$, then $\omega(t) = u(p, t)$ (temperature) or $\omega(t) = \partial_n u(p, t)$ (heat transfer) are usual observations/readings of the solution on the boundary. Thus we want to recover A or at least its spectrum $\sigma_A = \{\lambda_n\} \subset \mathbb{C}$ from the observation mapping

$$u(0) \to \omega(t).$$

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To do so, although we do NOT know A, we assume that it has a discrete spectrum $\{\lambda_n\} \subset \mathbb{C}$, and in general $\operatorname{Im}\lambda_n \to 0$ as $n \to \infty$, while $\operatorname{Re}\lambda_n \to -\infty$. If we denote its eigenfunctions by $\varphi_{n,0}$ and its associated eigenfunctions (roots) by $\varphi_{n,\nu}$ for $\nu = 1, ..., m_n - 1$, where m_n is the multiplicity of the eigenvalue λ_n, then we can write a formal solution to the evolution equation

$$u(t) = \sum_{n\geq 1} e^{\lambda_n t} \sum_{\nu=0}^{m_n-1} c_{n\nu}(f)\, p_{n\nu}(t)\varphi_{n\nu} \tag{2}$$

where the Fourier coefficients are $c_{n\nu}(f) = \langle f, \psi_{n\nu}\rangle$ and $\{\psi_{n\nu}\}$ is the biorthogonal system to $\{\varphi_{n,\nu}\}$. Here $p_{n\nu}$ are polynomials generated by the multiplicity of the eigenvalue λ_n. The observation then is given by

$$\omega(t) = \sum_{n\geq 1} e^{\lambda_n t} \sum_{\nu=0}^{m_n-1} c_{n\nu}(f)\, p_{n\nu}(t)\langle\varphi_{n\nu}, g\rangle. \tag{3}$$

In the best case, when all $c_{n\nu}(f) \neq 0$ and $\langle\varphi_{n\nu}, g\rangle \neq 0$ then it is possible to evaluate/extract all the λ_n from the observation (2).
Open problems: i) How do you choose the initial condition f, so we can observe all $e^{\lambda_n t}$, that is all $c_{n\nu}(f) \neq 0$? We need to know something about the biorthogonal system $\{\psi_{n\nu}\}$.

ii) How do you choose the observation g so all $\langle\varphi_{n\nu}, g\rangle \neq 0$? We need to know something about the root functions $\{\varphi_{n,\nu}\}$.

iii) How smooth is the sum (2), so we can choose g? We need some information on the type of convergence in (2) so (3) holds.

iv) How do we extract the λ_n and their multiplicity from a given signal given by (3) in finite time? When λ_n are complex values and the sum contains polynomials in t, it is much harder than the real case.

v) Find the best f and g that allow the identification of A by using the smallest number of observations. Evolution equations are often found in control theory, and for that purpose, we need finite number of observations done in finite time.

4. Marina Chugunova

4.1. **Computations of the instability index for a non-self-adjoint operators.** The stability of steady states is a basic question about the dynamics of any partial differential equation that models the evolution of a physical system.

In order to numerically evaluate the instability index of a given differential operator A, its computation should be reduced to a problem of linear algebra. Particularly for problems with periodic boundary conditions, it seems natural to restrict the operator A to a finite-dimensional space of trigonometric polynomials.
Open problem: Under what conditions the instability index (the total number of unstable eigenvalues) can be computed from the resulting finite dimensional matrix?

One difficulty is that the entries of the infinite matrix corresponding to the differential operator A grow with the row and column index, so that any truncation is not a small perturbation.

If A is a self-adjoint semi-bounded differential operator of even order, then the instability index can be estimated by variational methods, or computed directly from the zeros of the corresponding Evans function.

Understanding the spectrum of a non-self-adjoint operator is a much harder problem. It is not at all obvious how to restrict the computation of its instability index to a finite-dimensional subspace, or how to even estimate its dimension. Furthermore, the numerical calculation of eigenvalues can be extremely ill-conditioned even in finite dimensions.

5. Michael Demuth

5.1. **Spectral radius and operator norm.** Let A be a bounded linear operator on a Banach space X. Its spectral radius is defined by

$$\operatorname{spr}(A) := \max\{|z| \,:\, z \in \sigma(A)\}.$$

Gelfand proved the classical formula

$$\operatorname{spr}(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

Obviously, $0 \le \operatorname{spr}(A) \le \|A\|$. The question arises: What is the gap between $\|A\|$ and $\operatorname{spr}(A)$? Introduce the denotation

$$\operatorname{gap}(A) := \|A\| - \operatorname{spr}(A).$$

Open problems: i) For which class of operators holds

$$\operatorname{gap}(A) > 0 \text{ or } \operatorname{gap}(A) = 0,$$

respectively.

ii) What is the smallest $m \in (0, 1]$, such that

$$\operatorname{spr}(A) \le m\|A\|$$

or

$$\operatorname{gap}(A) \ge (1 - m)\|A\| \quad ?$$

Example 1. Let $X = \ell^1(\mathbb{N})$ and A be the weighted shift-operator defined according to the canonical standard basis by the infinite matrix

$$\begin{pmatrix} 0 & & & & & \\ b_1 & 0 & & & & \\ & b_2 & 0 & & & \\ & & b_1 & 0 & & \\ & & & b_2 & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}$$

where $b_1, b_2 > 0$ and $b_1 b_2 = 1$. In this case $\|A\| = \max\{b_1, b_2\}$ and $\sigma(A) = \{z \in \mathbb{C} : |z| \le 1\}$ and therefore $\operatorname{spr}(A) = 1$. Thus

- $\operatorname{gap}(A) = 0$: If $b_1 = b_2 = 1$ then $\|A\| = \operatorname{spr}(A)$.
- $\operatorname{gap}(A) > 0$: If $b_1 \ne b_2$ then $\|A\| > \operatorname{spr}(A)$.

This kind of estimates are useful in the following situation. Let K be a compact perturbation of A. Study the discrete spectrum of $B := A + K$. We are able to analyze the moments and the number of eigenvalues of B outside a ball of radius $\|A\|$. It is more interesting and also natural to enlarge this region up to the complement of a ball with radius $\operatorname{spr}(A)$.

6. Mark Embree

6.1. **Davies' conjecture about approximate diagonalization.** Consider a non-normal matrix $A \in \mathbb{C}^{n\times n}$. Define

$$\underline{s}(A, \varepsilon) := \inf_{\substack{\Delta, V \\ V^{-1}(A+\Delta)V \text{ diagonal}}} \|V\|\,\|V^{-1}\|\varepsilon + \|\Delta\|.$$

Open problem: Prove Davies' conjecture (2007): There exists a constant $C_n > 0$, independent of $A \in \mathbb{C}^{n\times n}$, such that $\underline{s}(A, \varepsilon) \le C_n\sqrt{\varepsilon}$.

It is known that the conjecture holds for Jordan blocks (then $C_n = 2$ suffices) and for 3×3 matrices with $\|A\| \le 1$ (then $C_n = 4$ suffices).

6.2. **Crouzeix' conjecture about the norm of matrix functions.** Let A be a bounded linear operator. It is known that $\|A^k\| \le 2 \max_{z\in W(A)} |z^k|$; $W(A)$ denotes the numerical range of A.

Open problem: Prove Crouzeix' conjecture: There exists a constant $C \ge 2$ such that for all analytic functions $f : W(A) \to \mathbb{C}$ holds $\|f(A)\| \le C \max_{z\in W(A)} |f(z)|$.

Crouzeix conjectured further that $C \le 11.08$.

7. Rupert L. Frank

7.1. **Laptev-Safronov conjecture.** Consider a Schrödinger operator $-\Delta + V$ in $\mathbb{R}^d$ with a complex potential V.

Open problems: i) What is the largest p such that all non-real eigenvalues lie in a disk around 0 of radius $D(\int_{\mathbb{R}^d} |V|^p \, dx)^{(p-d/2)^{-1}}$ (the constant $D > 0$ shall not depend on V)?

ii) What happens to embedded eigenvalues of self-adjoint Schrödinger operators under non-self-adjoint perturbations?

Both problems are related to the Laptev-Safronov conjecture which states that for every $d \in \mathbb{N}$ and $0 < \gamma \leq d/2$ there exists a constant $D_{\gamma,d} > 0$ such that for every potential V every non-real eigenvalue λ satisfies

$$|\lambda|^\gamma \leq D_{\gamma,d} \int_{\mathbb{R}^d} |V|^{\gamma + \frac{d}{2}} \, dx;$$

here $\gamma = p - d/2$ with p from problem i). The Laptev-Safronov conjecture is known to be true in dimension $d = 1$ if $\gamma = 1/2$ and in dimension $d \geq 2$ if $0 < \gamma \leq 1/2$.

8. Marcel Hansmann

8.1. **Tensor trick for perturbed operators.** Let A be a bounded self-adjoint operator in a Hilbert space and let K be a perturbation which is of trace class. Assume that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\sum_{\lambda \in \sigma_d(A+K)} \operatorname{dist}(\lambda, \sigma(A))^{1+\varepsilon} \leq C(\varepsilon) \|K\|_{1+\varepsilon}^{1+\varepsilon}.$$

Open problem: Does it follow that there exists $C > 0$ such that

$$\sum_{\lambda \in \sigma_d(A+K)} \operatorname{dist}(\lambda, \sigma(A)) \leq C \|K\|_1?$$

9. Michael Hitrik

9.1. **Upper bounds on the norm of the resolvent.** It is well known that the spectrum of a non-self-adjoint operator does not control its resolvent and that the latter may become very large far from the spectrum. Some general upper bounds on resolvents are provided by the abstract operator theory, and restricting the attention to the setting of semiclassical operators on $\mathbf{R}^n$, let us give a rough statement of such bounds. Assume that $P = p^w(x, hD_x)$ is the semiclassical Weyl quantization on $\mathbf{R}^n$ of a nice symbol p with $\operatorname{Re} p \geq 0$, say. Then the norm of the resolvent of P is bounded from above by a quantity of the form $\mathcal{O}(1) \exp(\mathcal{O}(1) h^{-n})$, provided that $z \in \operatorname{neigh}(0, \mathbf{C})$ is not too close to the spectrum of P. On the other hand, the available lower bounds on the resolvent of P, in the interior of the range of the symbol, coming from the pseudospectral considerations, are typically of the form $C_N^{-1} h^{-N}$, $N \in \mathbf{N}$, or $(1/C)e^{1/(Ch)}$, provided that p enjoys some analyticity properties, [DSZ04]. There appears to be therefore a substantial gap between the available upper and lower bounds on the resolvent, especially when $n \geq 2$, which, to the best of my knowledge, has so far only been bridged in the very special case of elliptic quadratic differential operators, see [HSV13].

Open problem: Is the upper bound sharp (especially for dimension $n \geq 2$)?

10. David Krejčiřík

10.1. **Semiclassical pseudomodes of Schrödinger operators with discontinuous potentials.** For smooth potentials there exists a quite general theory on the construction of semiclassical pseudomodes, see [DSZ04].

Open problem: Can the technique be adapted to discontinuous potentials?

In my joint paper with Henry [HK15] we have a non-trivial pseudospectrum in a toy model (complex Heaviside-type potential). However, our technique is restricted to the particular situation and the non-trivial pseudospectrum is rather generated by the behavior of the potential at infinity.

11. Michael Levitin

11.1. **Complex eigenvalues of indefinite pencil.** For $0 < c < 2$ define the $2n \times 2n$ matrices

$$A := \begin{pmatrix} c & 1 & & & \\ 1 & c & 1 & & \\ & 1 & c & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & c \end{pmatrix}, \quad B := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

where the numbers of 1 and -1 in B coincide (equal to n). The eigenvalues of the pencil $\lambda \mapsto A - \lambda B$ are the eigenvalues of $B^{-1}A = BA$. The spectrum is symmetric with respect to both $\mathbb{R}$ and $i\mathbb{R}$. The non-real eigenvalues are contained in the union of the two closed disks $\{\lambda \in \mathbb{C} : |\lambda \pm c| \leq 2\}$ whereas numerical examples suggest that they lie in their intersection.
Open problem: Prove that $|\lambda - c| \leq 2$ and $|\lambda + c| \leq 2$ holds for all $\lambda \in \sigma(BA)\backslash\mathbb{R}$.

11.2. **Indefinite Sturm-Liouville.** Recently, we have proved some conjectures related to the generalized eigenvalue problem $(-\frac{d^2}{dx^2} + \frac{c}{1+|x|})\psi = \lambda \mathrm{sgn}(x)\psi$.
Open problem: Replace the potential term by a more general function.

12. Marco Marletta

12.1. **Zeros and poles of Nevanlinna functions.** Let $m_1, m_2, \ldots$ be an infinite sequence of meromorphic functions with Im $m_j(\lambda) > 0$ if Im $\lambda > 0$ and Im $m_j(\lambda) < 0$ if Im $\lambda < 0$ (Nevanlinna functions). Suppose that there exists a non-empty interval $I \subseteq \mathbb{R}$ such that for every non-empty subinterval $J \subseteq I$ holds

$$\lim_{j\to\infty} \#\{\text{pole of } m_j \text{ in } J\} = \infty.$$

Let g be a function which is analytic in a complex neighborhood of I.
Open problems: i) Show that for every open complex neighborhood U of J,

$$\lim_{j\to\infty} \#\{\text{zero of } (m_j - g) \text{ in } U\} = \infty.$$

ii) Show that there exists a constant $C > 0$, independent of j, such that for every open complex neighborhood U of J,

$$\big|\#\{\text{zero of } (m_j - g) \text{ in } U\} - \#\{\text{pole of } m_j \text{ in } U\}\big| \leq C.$$

The result is known to hold if $\mu(J) := \lim_{j\to\infty} j^{-1}\, \#\{\text{pole of } m_j \text{ in } J\}$ exists.

13. Boris Mityagin

13.1. **Schauder basis for Schrödinger operators with periodic boundary conditions.** Consider the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + V$ on $(0, \pi)$ with periodic boundary conditions, where the potential V is a trigonometric polynomial, i.e.,

$$V(x) = \sum_{k=-m}^{m} v_k e^{2ikx}; \quad v_k \in \mathbb{C},\ |k| \leq m.$$

Open Problem: For which sets of coefficients $\{v_k\}_{k=-m}^{m}$ do the eigenfunctions form a Schauder basis for $L^2(0, \pi)$?

Known Case: Let $V(x) := e^{-2ix} + be^{2ix}$. Then the answer is "yes" if and only if $|b| = 1$.

14. Kwang Shin

14.1. Non-polynomial complex potentials. Consider

$$H = -\frac{d^2}{dx^2} + x^m + a_1 x^{m-1} + \cdots + a_m$$

in $L^2(\mathbb{R}_+)$ with $y(0)\cos\theta + y'(0)\sin\theta = 0$, where $a_j \in \mathbb{C}$ and $\theta \in \mathbb{C}$. It is known that H has infinitely many eigenvalues. Moreover, all eigenvalues are real if and only if $a_j \in \mathbb{R}$ for all j and $\theta \in \mathbb{R}$.
Open Problem: Is there any non-self-adjoint non-polynomial potential case H, either in $L^2(\mathbb{R})$ or in $L^2(\mathbb{R}_+)$, that generates infinitely many real eigenvalues and at most finitely many non-real eigenvalues?

15. Petr Siegl

15.1. Riesz basis for Schrödinger operator with complex potential. Consider the self-adjoint harmonic oscillator $A_0 := -\mathrm{d}^2/\mathrm{d}x^2 + x^2$ in $L^2(\mathbb{R})$. Define $A := A_0 + V$ with a complex-valued potential $V \in L^\infty(\mathbb{R})$.
Open problem: Is the eigensystem of A a Riesz basis?

It is known that the answer is 'yes' if $V \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$.

16. David A. Smith

16.1. Spectral representation of two-point differential operators. Augmented eigenfunctions are a class of spectral functionals which have been shown to be useful in expressing solutions of initial-boundary value problems [FS15, PSss, Smi14]. This is particularly important in the case where the spatial differential operator is degenerate irregular in the sense of Locker [Loc08], as no other effective solution representation is known. They also provide a spectral theorem where the inverse of the operator is diagonalized.
Open problems: i) Are there other applications for augmented eigenfunctions?

ii) Can a spectral theory be developed using augmented eigenfunctions?

*Other Open Problems

17. Lyonell Boulton

17.1. **Numerical approximation of rigorous enclosures for the spectrum of J-self-adjoint operators.** The aim of this theme is to device strategies for computing rigorous (hopefully sharp) bounds/enclosures for the spectrum of J-self-adjoint operators by means of projected space methods. The theory of computation for spectra of self-adjoint operators is classical and well developed. J-self-adjoint operators share many properties with their self-adjoint counterpart. I am interested in discussing to what extent general strategies for numerically estimating spectra of J-self-adjoint operators can be devised.
Open problems: i) Computation of rigorous rough enclosures for J-self-adjoint operators, taking into account the structure of the conjugation J into the projection scheme.
ii) Computation of sharp (numerically relevant) enclosures in specific or generic cases.
iii) Impact in the study of evolution problems for J-self-adjoint operators.

18. Tanya J. Christiansen

18.1. **Isoresonant potentials.** Consider the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^d$, where the potential $V \in L^\infty_0(\mathbb{R}^d)$. If $V \in C^\infty_c(\mathbb{R}^d;\mathbb{R})$, then if V is non-trivial the Schrödinger operator has infinitely many resonances. However, if $d \geq 2$ there are non-trivial complex-valued potentials $V \in C^\infty_c(\mathbb{R}^d)$ for which the corresponding Schrödinger operator has no resonances. More generally, one can explicitly construct families of isoresonant, compactly supported complex-valued potentials in dimensions at least 2.
Open problems: Is there some other data related in some way to spectral or pseudo spectral properties of the operators that distinguish elements (potentials) in these sets? There are related families of isospectral Schrödinger operators in other settings– on the unit circle, for example. One can ask the same question there.

19. Michael Demuth

19.1. **Estimates for the resolvent near the spectrum.** Let A be a linear operator on a Banach space. Let K be a compact perturbation of A. The approximation numbers of K are defined by

$$\alpha_N(K) := \inf\{\|K - F\|\,,\, \mathrm{rank}(F) < N\}.$$

We consider only compact operators K with $\lim_{N\to\infty} \alpha_N(K) = 0$.

The objective is to estimate the numbers of eigenvalues of the perturbed operator $B := A + K$ in certain regions of the complex plane.

Let $\Omega_t = \{\lambda \in \mathbb{C}\,,\, |\lambda| > t\}$. Denote $\mathrm{spr}(A) := \max\{|\lambda|\,,\, \lambda \in \sigma(A)\}$ and assume $\mathrm{spr}(A) < t < s$. Denote by $n_B(s)$ the number of eigenvalues of B in Ω_s. In [DHHK15] we obtained

$$n_B(s) \leq \frac{(2e)^{\frac{p}{2}}}{\log \frac{s}{t}} \frac{\sup_{\lambda\in\Omega_t} \|(\lambda - A)^{-1}\|^p}{\left(1 - \alpha_{N+1}(K) \sup_{\lambda\in\Omega_t} \|(\lambda - A)^{-1}\|\right)^p} \sum_{j=1}^{N} \left(\alpha_{N+1}(K) + \alpha_j(K)\right)^p . \tag{4}$$

Here N has to be so large that

$$\alpha_{N+1}(K) \sup_{\lambda\in\Omega_t} \|(\lambda - A)^{-1}\| < 1.$$

The optimal result depends on the behavior of $\|(\lambda - A)^{-1}\|$ near the spectrum of A, i.e. on Ω_s. This is typical for many spectral considerations. It is also related to the pseudospectrum of A. For instance if $|\lambda| > \|A\|$ then

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{|\lambda| - \|A\|}$$

and therefore (4) becomes

$$n_B(s) \leq \frac{(2e)^{\frac{p}{2}}}{\log\frac{s}{t}\,(t-(\|A\|+\alpha_{N+1}(K)))^p}\sum_{j=1}^{N}(\alpha_{N+1}(K)+\alpha_j(K))^p\,.$$

Open problems: i) Classify the operators for which the resolvent is polynomially bounded if $\lambda \to \sigma(A)$?

ii) Classify the operators for which one can find an $M \geq 1$ such that

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{\operatorname{dist}(\lambda, \sigma(A))}$$

for all $\lambda \in \operatorname{res}(A)$ or

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda| - \operatorname{spr}(A)}$$

for $|\lambda| > \operatorname{spr}(A)$. Remark: For instance in Hilbert spaces, the bound in ii) is true for normal operators with $M = 1$.

20. Michael Hitrik

20.1. **Inverse spectral problems for non-self-adjoint operators, especially in the semiclassical limit.** Given a suitable h-pseudodifferential operator $P = p^w(x, hD_x)$ on $\mathbf{R}^n$ or a compact manifold, we would like to understand what information about the classical symbol p can be determined from the spectrum of P, in the semiclassical limit $h \to 0$. We are especially interested in cases when P is non-self-adjoint, with the inverse problems for resonances and for damped wave equations being important sources of motivation. See [DH12], [Hal13], [Pha] for some of the recent works on semiclassical inverse spectral problems in the non-self-adjoint setting.

20.2. **Spectra for non-self-adjoint operators in the presence of symmetries.** The proof of the reality of the exponentially small eigenvalues of the Kramers-Fokker-Planck type operators in [HHS11] depends on a reflection symmetry for such operators, and there are many natural non-self-adjoint situations where symmetries play a role, including PT-symmetric operators and operators with supersymmetric structures. See also [Shi02], [KS02].

21. David Krejčiřík

21.1. **Large-time behavior of the heat equation: subcriticality versus criticality.** This open problem is a repetition of the open problem raised during previous meetings in Prague (2010) and Barcelona (2012)

`http://www.ujf.cas.cz/ESFxNSA/`
`http://gemma.ujf.cas.cz/~david/OTAMP2012/OTAMP2012.html`

but little progress has been made so far. Please visit the links above for more details and references.

Our conjecture is that the solutions of the heat equation "decay faster" for large times provided that the generator is "more positive" in the sense of the validity of a Hardy-type inequality. There exist both semigroup (with Zuazua [KZ10]) and heat-kernel (with Fraas and Pinchover [FKP10]) versions of the conjecture and the latter involves non-self-adjoint operators too. The conjecture has been supported by several particular situations, but there exists no general result yet.

In the self-adjoint case, the conjectures can be stated as follows. Let Ω be an open connected subset of $\mathbb{R}^d$. Let H_0 and H_+ be two self-adjoint operators in $L^2(\Omega)$ such that $\inf\sigma(H_0) = \inf\sigma(H_+) = 0$. Assume that H_+ is *subcritical*, in the sense that there is a smooth positive function $\rho : \Omega \to \mathbb{R}$ such that $H_+ \geq \rho$ (Hardy inequality). On the other hand, H_0 is assumed to be *critical*, in the sense that $\inf\sigma(H_0 - V) < 0$ for any non-negative non-trivial $V \in C_0^\infty(\Omega)$.

Conjecture 1 (Semigroup version, [KZ10])**.** There is a positive function (weight) $w : \Omega \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{\left\|e^{-H_+ t}\right\|_{L^2_w(\Omega)\to L^2(\Omega)}}{\left\|e^{-H_0 t}\right\|_{L^2_w(\Omega)\to L^2(\Omega)}} = 0\,.$$

Conjecture 2 (Heat-kernel version, [FKP10]). Let H_+ and H_0 be in addition elliptic differential operators whose coefficients satisfy some minimal regularity assumptions so that the heat kernels exist. Then

$$\lim_{t\to\infty} \frac{e^{-H_+ t}(x,x')}{e^{-H_0 t}(x,x')} = 0$$

locally uniformly for $(x,x') \in \Omega \times \Omega$.

Open problem: Prove the conjectures or find a counterexample.

21.2. **The cloaking effect in metamaterials: beyond ellipticity.** In mathematical models of metamaterials characterized by negative electric permittivity and/or negative magnetic permeability, there appear operators of the type

$$\operatorname{div} \operatorname{sgn} \operatorname{grad},$$

where $\operatorname{sgn}(x) = \pm 1$ for $x \in \Omega_\pm$, two disjoint subsets of $\mathbb{R}^d$ divided by a smooth hypersurface.
Open problem: How to define such an operator as a self-adjoint operator in an L^2 setting?

There exist numerous works on a changed problem in which there is a small complex constant added to the minus one in the sign function (making the problem non-self-adjoint, but sectorial). The only exception (apart from the one-dimensional situation, which is elementary) seems to be my recent joint paper with Behrndt [BK]. Here we solve the original problem for a particular geometry (rectangle) with help of a refined extension theory. It turns out that the domain of the self-adjoint operator is not a subset of the Sobolev space H^1 and there is an essential spectrum (although the geometry is bounded).

22. Rudi Weikard

22.1. **Open problem for complex-valued periodic potentials.** If q is a complex-valued periodic potential with period a consider the differential expression $-y'' + qy$. For x_0 varying in $[0, a]$ introduce the solutions $c(\cdot, x_0, \lambda)$ and $s(\cdot, x_0, \lambda)$ of $-y'' + qy = \lambda y$ satisfying the initial conditions $c(x_0, x_0, \lambda) = s'(x_0, x_0, \lambda) = 1$ and $c'(x_0, x_0, \lambda) = s(x_0, x_0, \lambda) = 0$.

The periodic and semi-periodic eigenvalues are given as the zeros of the entire function $(\operatorname{tr} \mathcal{M})^2 - 4$ where $\mathcal{M}$ is the monodromy operator associated to q. In fact $\operatorname{tr} M(\lambda) = c(x_0 + a, x_0, \lambda) + s'(x_0 + a, x_0, \lambda)$ which is, in fact, independent of x_0.

The Dirichlet and Neumann eigenvalues with respect to the interval $[x_0, x_0 + a]$ are given as the zeros of the entire functions $s(x_0 + a, x_0, \cdot)$ and $c'(x_0 + a, x_0, \cdot)$, respectively, and depend, in general, on x_0.

Let $d(\lambda)$, $p(x_0, \lambda)$, and $r(x_0, \lambda)$ denote the multiplicities of λ as zeros of $(\operatorname{tr} \mathcal{M})^2 - 4$, $s(x_0 + a, x_0, \cdot)$ and $c'(x_0 + a, x_0, \cdot)$, respectively (these are also the algebraic multiplicities of the corresponding eigenvalues). Moreover, let $p_i(\lambda) = \min\{p(x_0, \lambda) : x_0 \in [0, a]\}$ and $r_i(\lambda) = \min\{r(x_0, \lambda) : x_0 \in [0, a]\}$.

One has then the following facts (see [GW96]):

1. $p_i(\lambda) = r_i(\lambda)$.
2. If $d(\lambda) > 0$, $p(x_0, \lambda) > 0$, and $r(x_0, \lambda) > 0$, then $p_i(\lambda) = r_i(\lambda) > 0$.
3. $d(\lambda) - p_i(\lambda) - r_i(\lambda) \geq 0$.

Note that a (semi-)periodic eigenvalue λ has geometric multiplicity 2 if and only if it is both a Dirichlet and a Neumann eigenvalue. If λ is a point where two linearly independent Floquet solutions do not exist, i.e., a (semi-)periodic eigenvalue with geometric multiplicity 1, then $d(\lambda) > 0$ and $p_i(\lambda) = r_i(\lambda) = 0$ so that $d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0$. In the self-adjoint case this is the only way to make $d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0$.
Open problem: Now the question arises whether in the non-self-adjoint case it is possible that $d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0$ when λ is a (semi-)periodic eigenvalue of geometric multiplicity 2. Thus, either prove that $d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0$ implies $p_i(\lambda) + r_i(\lambda) = 0$ or give an example to the contrary.

References

[BK] J. Behrndt and D. Krejčiřík. An indefinite Laplacian on a rectangle. *J. Anal. Math. (to appear)*. arXiv: 1407.7802, `http://arxiv.org/abs/1407.7802`.

[DH12] K. Datchev and H. Hezari. Resonant uniqueness of radial semiclassical Schrödinger operators. *Appl. Math. Res. Express. AMRX*, (1):105–113, 2012.

[DHHK15] M. Demuth, F. Hanauska, M. Hansmann, and G. Katriel. Estimating the number of eigenvalues of linear operators on Banach spaces. *J. Funct. Anal.*, 268(4):1032–1052, 2015.

[DSZ04] N. Dencker, J. Sjöstrand, and M. Zworski. Pseudospectra of semiclassical (pseudo-) differential operators. *Commun. Pure Appl. Math.*, 57:384–415, 2004.

[FKP10] M. Fraas, D. Krejčiřík, and Y. Pinchover. On some strong ratio limit theorems for heat kernels. *Discrete Contin. Dynam. Systems A*, 28:495–509, 2010.

[FS15] A. S. Fokas and D. A. Smith. Evolution PDEs and augmented eigenfunctions. I finite interval. arXiv:1303.2205 `http://arxiv.org/abs/1303.2205`, 2015.

[GW96] F. Gesztesy and R. Weikard. Picard potentials and Hill's equation on a torus. *Acta Math.*, 176(1):73–107, 1996.

[Hal13] M. A. Hall. Diophantine tori and non-selfadjoint inverse spectral problems. *Math. Res. Lett.*, 20(2):255–271, 2013.

[HHS11] F. Hérau, M. Hitrik, and J. Sjöstrand. Tunnel effect and symmetries for Kramers-Fokker-Planck type operators. *J. Inst. Math. Jussieu*, 10(3):567–634, 2011.

[HK15] R. Henry and D. Krejčiřík. Pseudospectra of the Schroedinger operator with a discontinuous complex potential. arXiv: 1503.02478, 2015.

[HSV13] M. Hitrik, J. Sjöstrand, and J. Viola. Resolvent estimates for elliptic quadratic differential operators. *Analysis & PDE*, 6:181–196, 2013.

[KS02] M. Klaus and J. K. Shaw. Purely imaginary eigenvalues of Zakharov-Shabat systems. *Phys. Rev. E (3)*, 65(3):036607, 5, 2002.

[KZ10] D. Krejčiřík and E. Zuazua. The Hardy inequality and the heat equation in twisted tubes. *J. Math. Pures Appl.*, 94:277–303, 2010.

[Loc08] J. Locker. Eigenvalues and completeness for regular and simply irregular two-point differential operators. *Mem. Amer. Math. Soc.*, 195(911):viii+177, 2008.

[Pha] Q. S. Phan. Spectral monodromy of non selfadjoint operators. arXiv: 1303.1352, `http://arxiv.org/abs/1303.1352`.

[PSss] B. Pelloni and D. A. Smith. Evolution PDEs and augmented eigenfunctions. II half line. *J. Spec. Theory*, 2015 (in press).

[Shi02] K. C. Shin. On the Reality of the Eigenvalues for a Class of $\mathcal{PT}$-Symmetric Oscillators. *Comm. Math. Phys.*, 229:543–564, 2002.

[Smi14] D. A. Smith. The unified transform method for linear initial-boundary value problems: a spectral interpretation. arXiv:1408.3659, `http://arxiv.org/abs/1408.3659`, 2014.