Mathematical aspects of physics with non-self-adjoint operators

organized by
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Workshop Summary

Overview and focus

The workshop, sponsored by the AIM and the NSF, examined state-of-the-art techniques for the mathematically rigorous analysis of non-self-adjoint phenomena encountered in main stream and newly developing fields of physics. Concrete problems for linear operators and pencils were addressed as described below.

The considered open problems were separated into five categories with some natural intersections.

- Spectrum, resonances and pseudospectra.
- Basis properties of eigenvectors.
- Microlocal and semiclassical methods.
- Numerical computation of spectra.
- Inverse problems for resonances.

Morning talks, discussion sessions and group work roughly followed this classification.

There were 14 attendants from the US and 9 from Europe.

Morning talks

Each morning we had two introductory lectures.

Monday, June 8

(1) **M. Levitin**: *Spectral problems for linear pencils*
The talk focused on two spectral problems for linear pencils, an indefinite Sturm-Liouville pencil and a Jacobi matrix model. Despite apparent simplicity of these models, generalizations of established results on localization of eigenvalues for particular potentials and matrix entries turned out to be difficult. Both models were suggested as open problems for group work. The matrix model was selected.

(2) **M. Embree**: *Functions of Nonnormal Matrices and the Behavior of Dynamical Systems*
A survey on numerical challenges and techniques for non-normal and also non-linear problems was given in this talk. A number of open problems from the literature was highlighted. These included: Davies's conjecture about approximate diagonalization, Crouzeix's conjecture (suggested as an open problem for group works) about the norm of matrix functions, the inverse numerical range problem, the question of "Do

pseudospectra determine behavior?" and the role of nonnormality in the Lyapunov matrix equation.

Tuesday, June 9

(1) **M. Hansmann**: *Lieb-Thirring type estimates for non-selfadjoint operators*
(2) **R. Frank**: *Eigenvalues of Schrödinger operators with complex potentials*
These two talks were closely linked. The speakers presented two different perspectives on new exciting results concerning Lieb-Thirring type inequalities and bounds on individual eigenvalues for non-selfadjoint operators, particularly Schrödinger operators with complex potential. Two related open problems (the tensor trick and the Laptev-Safronov conjecture) were suggested and selected for group works.

Wednesday, June 10

(1) **M. Hitrik**: *Subelliptic estimates and semigroup smoothing for quadratic operators*
Ideas on new simplified proofs of the global subelliptic estimates for non-selfadjoint quadratic differential operators were examined in this talk. One of the interesting features highlighted was the fact that the method allows determination of accurate smoothing estimates for the associated heat semigroup in the small time limit regime.
(2) **M. Shubov**: *Mathematical Analysis of High Aspect-Ratio Aircraft Wing in Subsonic Air Flow: Incompressible and Compressible Cases*
A survey on applications to fluttering of aircraft wings, modeled by linear non-selfadjoint integro-differential operators was presented in this talk. A series of related results on asymptotic, spectral, and stability analysis was examined.

Thursday, June 11

(1) **T. Christiansen**: *An introduction to resonances*
This speaker delivered a survey on the state-of-the-art of resonance analysis for Schrödinger operators with an emphasis in odd dimensions. The behavior for real and complex potentials was compared and a number of open problems on the counting number of resonances near the spectrum for the complex case were highlighted.
(2) **Y. Almog**: *On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent*
In this talk a few open problems for non-selfadjoint Schrödinger operators arising in model from superconductivity were discussed. A particular emphasis was made on non-emptiness of the spectrum and completeness of the corresponding eigensystem.

Friday, June 12

(1) **S. Bögli**: *Convergence of pseudospectra : Can the resolvent norm be constant on an open set?*
Recent exciting results about non-standard behavior of pseudospectra in Banach spaces were discussed in this talk. These include the phenomenon of constant resolvent norm on a non-empty open set. New results on global minima of the resolvent norm and impact on convergence properties of the pseudospectra were also announced.

(2) **R. Weikard**: *Inverse problems for resonances*

This speaker presente a survey on the state-of-the-art for the analysis of one-dimensional inverse problems for resonances of Schrödinger operators with complex potentials. Key conditions for the potential include the so-called "first momentum condition". A few open problems were proposed including the setting of Jacobi matrices.

Group work and afternoon activities

During the first afternoon twelve open problems were suggested in an open problem session moderated by Dencker. After a pre-selection made on Monday evening, the participants picked four open problems and separated into groups on Tuesday. Discussions on these four problems (listed below) was led by Almog, Hansmann, Krejčiřík and Levitin. On Wednesday, the groups reported on progress. The group led by Levitin decided to dissolve and merge into the other three. Substantial progress was made by Mityagin towards the solution of the problem *Schauder bases of periodic functions and multipliers* posed by Boulton. This was reported on Wednesday and the problem was further developed. Work groups continued on Wednesday with reports on Thursday. A further open problem by Frank was added in order to split the attendants into four groups again. On Friday there were final reports on the group work, a few more open problems were added to the final list and Mityagin delivered a full account on his progress.

Selected problems (full statements in the List of Open Problems):

(1) **Y. Almog**: *Completeness of eigensystems for Schrödinger operators with complex potentials*

Attempts were made to apply various methods in order to determine completeness of the eigensystem for the operator

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + i|x|^\alpha$$

around the critical case $\alpha = 2/3$. At present it is known that for $\alpha > 2/3$ there is completeness, but the question remains open for $0 < \alpha \leq 2/3$. Several modifications of the question were suggested. The most promising involves a transformation of the form $x = z^\beta$ where β is to be determined.

(2) **R. Frank**: *The Laptev-Safronov conjecture*

A few ideas on how to formulate more concrete questions on this problem were considered. These include:

- If $d = 1$, is the "conjecture" true for $\gamma > 1/2$?
- Can this be studied by means of Wigner-von Neumann potentials?
- Are there quantitative inverse spectral theory results?
- What is the behavior of embedded eigenvalues of a self-adjoint Schrödinger operator under a dissipative potential perturbation?
- Can counterexamples be obtained using Wigner-von Neumann potentials? In general dimensions, for $\gamma > 1/2$, can counterexamples be obtained using Ionescu-Jerison potentials?

(3) **M. Hansmann**: *Tensor trick for perturbed operators*

The discussion led to the modification of the original question to the current version, which is more likely to have an affirmative answer. In the simplest settings (e.g. for 2×2 matrices), the latter was verified.

(4) **D. Krejčiřík**: *Pseudomodes for Schrödinger operators with discontinuous potential*
Pseudomodes were constructed (Dencker and Hitrik) in the semiclassical one-dimensional setting, i.e. for $-h^2\Delta + \mathrm{i}\, V(x)$, $h > 0$, with $V(x) = \operatorname{sign}(x)$ or Hölder continuous V. Pseudospectrum of the toy model $-\Delta + \mathrm{i} \operatorname{sign}(x)$ was computed by Embree. and multi-dimensional generalizations of the toy model were suggested by Dencker, e.g. for $d = 2$, consider a piecewise constant V with a jump supported on a line or curve.

(5) **M. Levitin**: *Complex eigenvalues of indefinite pencil*
A few numerical experiments were performed. The group concurred that the preliminary results are promising. They agreed to report on the matter in due course.

Abstracts of morning talks

Monday, June 8

(1) **M. Levitin**: *Spectral problems for linear pencils*
My principal interests related to the workshop are in spectral problems for linear pencils $A - \lambda B$ where both coefficients A, B are self-adjoint sign-indefinite operators. In this case, the pencil spectra are generally speaking non-real, and localization of eigenvalues and their asymptotic behavior (with respect to some parameters which may be present in the problem) lead to difficult and interesting problems combining operator theory, complex analysis, special functions and many other topics.

I will pose some open problems arising in two realizations of the pencil - a matrix operator first studied by Davies and Levitin (2014), with interesting challenges in its random analogues, and an indefinite Sturm-Liouville pencil by Behrndt et al. for which partial results were obtained in Levitin-Seri (2015). Despite apparent simplicity of these models, they turn out to be remarkably deep and difficult.

(2) **M. Embree**: *Functions of Nonnormal Matrices and the Behavior of Dynamical Systems*
Nonnormality raises a series of issues that complicate the analysis of dynamical systems. This talk will explore several of these issues, including some open problems in the literature: Davies's conjecture about approximate diagonalization; Crouzeix's conjecture about the norm of matrix functions; the inverse numerical range problem of Malamud, Uhlig, and Carden; the question of "Do pseudospectra determine behavior", as explored by Ransford, Greenbaum, and Trefethen; and the role of nonnormality in the Lyapunov matrix equation.

We shall also discuss how best to generalize pseudospectra to more sophisticated dynamical systems (differential-algebraic equations, delay differential equations, nonlinear differential equations). In particular, we will show how the most natural generalization of pseudospectra for the generalized eigenvalue problem fails to describe transient growth in differential-algebraic equations, and we will propose a more suitable alternative.

References:

[CE-2012] R. L. Carden and M. Embree, http://dx.doi.org/10.1137/120872693*Ritz value localization for non-Hermitian matrices*, SIAM J. Matrix Analysis Appl., 33(4), 1320–1338, 2012.

[C-2077] M. Crouzeix, http://dx.doi.org/10.1016/j.jfa.2006.10.013*Numerical range and functional calculus in Hilbert space*, 244, 668–690, 2007.

[D-2077] E. B. Davies, http://dx.doi.org/10.1137/060659909*Approximate diagonalization*, SIAM J. Matrix Analysis Appl., 29(4), 1051–1064, 2007.

[R-2007] T. Ransford, http://dx.doi.org/10.1137/060658126*On pseudospectra and power growth*, SIAM J. Matrix Analysis Appl., 29(3), 699–711, 2007.

[TE-2005] L. N. Trefethen and M. Embree, *Spectra and Pseudospectra.* Princeton, 2005. See especially Chapters 25 and 47.

Tuesday, June 9

(1) **M. Hansmann**: *Lieb-Thirring type estimates for non-selfadjoint operators*

Consider a selfadjoint Schrödinger operator $H = -\Delta + V$ on $L^2(^d)$, where $d \geq 3$, with a potential $V :^d \to$ satisfying $|V(x)| \to 0$ for $|x| \to \infty$ (at least in some averaged sense). In this case the spectrum of H consists of $[0, \infty)$ and a possible set of negative discrete eigenvalues, which can accumulate at 0 only. The classical Lieb-Thirring inequalities provide information on the rate of convergence of these eigenvalues to zero, given more restrictive assumptions on V: they say that for $\gamma \geq 0$

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^\gamma \leq C_{\gamma,d} \int_d |V(x)|^{\gamma + \frac{d}{2}} \, dx.$$

In particular, if $n_H(-s)$, $s > 0$, denotes the number of discrete eigenvalues of H below $-s$, then

$$n_H(-s) \leq \frac{C_{\gamma,d}}{s^\gamma} \int_d |V(x)|^{\gamma + \frac{d}{2}} dx.$$

In recent times, starting with an article by Frank, Laptev, Lieb and Seiringer (2006), there has been an increasing interest in extending the above inequalities to the non-selfadjoint setting, that is, to the case where the potential V is complex-valued. Note that in this case the discrete eigenvalues of H can be complex and might (at least in principle) accumulate at every point in $[0, \infty)$.

In this talk I will try to present and discuss some of the results that have been obtained for this more general setting. As we will see, the situation here is not yet as clear as in the selfadjoint case and several open problems still need to be addressed.

Reference:

[DHMK-2013] M. Demuth, M. Hansmann, and G. Katriel, http://dx.doi.org/10.1007/978-3-0348-0591-9_2*Eigenvalues of non-selfadjoint operators: a comparison of two approaches*, Mathematical physics, spectral theory and stochastic analysis, 107163, Oper. Theory Adv. Appl., 232, Birkhuser/Springer Basel AG, Basel, 2013

(2) **R. Frank**: *Eigenvalues of Schrödinger operators with complex potentials*

First I would like to discuss bounds for single eigenvalues. Namely, the question is, given the L^p norm of the potential, what is the smallest region in the complex plane in which all the eigenvalues lie. In particular, is this region bounded. I will explain that there are thresholds $p = (d + 1)/2$ and $p = d$ for general and for radial potentials, respectively. This is, in part, based on joint work with B. Simon.

Secondly I would like to discuss bounds for sums of eigenvalues. The question is to find an analogue of the Lieb-Thirring inequality for complex potentials. These bounds limit, in particular, the accumulation of eigenvalues to the positive axis. I will present results up to the threshold $p = (d+1)/2$ obtained jointly with J. Sabin.

References:

[FS-2015] R. L. Frank and B. Simon, http://arxiv.org/abs/1504.01144*Eigenvalue bounds for Schrödinger operators with complex potentials. II*, arXiv:1504.01144.

[FS-2014] R. L. Frank and J. Sabin, http://arxiv.org/abs/1404.2817*Restriction theorems for orthonormal functions, Strichartz inequalities, and uniform Sobolev estimates*, arXiv:1404.2817v3.

Wednesday, June 10

(1) **M. Hitrik**: *Subelliptic estimates and semigroup smoothing for quadratic operators*

Using an approach based on the techniques of FBI transforms, we give a new and simplified proof of the global subelliptic estimates for non-selfadjoint quadratic differential operators, under a natural averaging condition on the Weyl symbols of the operators, due to K. Pravda-Starov. The loss of the derivatives in the subelliptic estimates depends directly on algebraic properties of the Hamilton maps of the symbols. Using the FBI point of view, we also give accurate smoothing estimates for the associated heat semigroup in the limit of small times. This is joint work with Karel Pravda-Starov and Joe Viola.

(2) **M. Shubov**: *Mathematical Analysis of High Aspect-Ratio Aircraft Wing in Subsonic Air Flow: Incompressible and Compressible Cases*

The goal of this talk is to present a series of results on asymptotic, spectral, and stability analysis of a mathematical model of a long slender aircraft wing in a surrounding subsonic airflow. In fact we consider two models: the case when the airflow is assumed to be incompressible and the case when the compressibility of the air is taken into account. Both models are linear, which is known to be acceptable in the subsonic regime.

Thursday, June 11

(1) **T. Christiansen**: *An introduction to resonances*

Mathematically, resonances may serve as a replacement for discrete spectral data for a class of operators with continuous spectrum. Physically, they correspond to decaying waves. Although there are a number of ways to define resonances, one is as the eigenvalues of a certain non-self-adjoint operator.

This talk focuses on resonances for Schrödinger operators with compactly supported potentials in odd dimensional Euclidean space, beginning with a definition. In particular we concentrate on upper and lower bounds on the number of resonances in a ball of radius r and the fundamental question of the existence of resonances for a nontrivial potential.

(2) **Y. Almog**: *On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent*

We consider the eigenspace of $\mathcal{P} : D \to L^2(\Omega)$, where

$$\mathcal{P} = -\sum_{k=1}^{d} e^{2i\alpha_k}(\partial_k - iA_k)^2 + V$$

In the above $A = (A_1, \dots, A_d)$ is a smooth magnetic potential and V is a complex potential. The domain is a smooth unbounded subset of $\mathbb{R}^d$, and D is a subset of $H_0^1(\Omega)$. We then apply the results to the linearized Ginzburg-Landau operator in a half-space.

Joint work with Bernard Helffer.

Friday, June 12

(1) **S. Bögli**: *Convergence of pseudospectra : Can the resolvent norm be constant on an open set?*

In this talk two closely related problems are addressed. First, we present a pseudospectral convergence result for unbounded linear operators converging in norm resolvent sense. A necessary assumption is that the limiting operator does not have constant resolvent norm on an open set. In the second part of the talk, we will focus on the question 'Can the resolvent norm be constant on an open set?'. We discuss the problem history and explain that if the answer is 'Yes', then the constant is the global minimum of the resolvent norm. This talk is based on joint work with Petr Siegl.

(2) **R. Weikard**: *Inverse problems for resonances*

In this talk inverse problems for the one-dimensional Schrödinger equation, i.e., the equation

$$-y'' + qy = \lambda y$$

are revisited. Here q is a complex-valued potential satisfying suitable integrability conditions. Two cases are considered: regular problems on a compact interval and the half-line problem.

Uniqueness for Borg's problem (two spectra given) in the case of an integrable potential on the interval $[0, 1]$ was established in 2001 in [MR1946191]. Levinson's problem (Dirichlet spectrum and Neumann data) and Marchenko's problem (Dirichlet spectrum and "norming" constants) are also treated in that paper. The stability problem for the two-spectra case was investigated in [**??**]. For this one assumes knowledge of N eigenvalues in either set up to a given error ε (finite noisy data) and arrives then at a statement of the form

$$\Big| \int_0^x (q(t) - \tilde{q}(t))dt \Big| \le C(\varepsilon \log N + N^{-1/2}).$$

For half-line problems we are interested in the inverse resonance problem, where (to define the resonances in the first place) we require q to be compactly supported. In this case the Jost solutions $f(x, k)$, viewed as functions of the momentum parameter k, are entire functions. Their zeros in the upper half-plane are related to eigenvalues (of which there are only finitely many), those in the lower half-plane to resonances. Location of eigenvalues and resonances determines a potential uniquely (under mild additional hypotheses) as was shown in [MR1994689] (2003). Note that there is no information on norming constants required (or available). The related stability question (again, finite noisy data) was established in [MR2594334] (2010). A heuristic argument showing that finite data may reasonably give pertinent information on the potential will also be presented.

References:

[MR1994689] B. M. Brown, I. Knowles, and R. Weikard. http://dx.doi.org/10.1112/S0024610703004654 *On the inverse resonance problem,* J. London Math. Soc. (2), 68(2):383–401, 2003.

[MR1946191] B. M. Brown, R. A. Peacock, and R. Weikard. http://dx.doi.org/10.1016/S0377-0427(02)00577-0 *A local Borg-Marchenko theorem for complex potentials*, J. Comput. Appl. Math., 148(1):115–131, 2002.

[MR2594334] M. Marletta, R. Shterenberg, and R. Weikard. http://dx.doi.org/10.1007/s00220-009-0928-8 *On the inverse resonance problem for Schrödinger operators*, Comm. Math. Phys., 295(2):465–484, 2010.

[MR2158108] M. Marletta and R. Weikard. http://dx.doi.org/ 10.1088/0266-5611/21/4/005 *Weak stability for an inverse Sturm-Liouville problem with finite spectral data and complex potential*, Inverse Problems, 21(4):1275–1290, 2005.