Nonstandard methods in combinatorial number theory

organized by
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Workshop Summary

The purpose of this workshop was to continue the use of nonstandard methods in combinatorial number theory and Ramsey theory. The organizers invited experts in nonstandard analysis, additive combinatorics, Ramsey theory, ergodic theory, and model theory with the hopes that this unique blend of experts would be able to influence each other in a positive way.

Every morning there were two lectures addressed to the entire workshop. Due to the afoementioned blend of fields of expertise, the organizers decided it would be a good idea to have introductory lectures given by members of the different groups. Introductions to nonstandard methods and their role in additive combinatorics and Ramsey theory were given by David Ross, Renling Jin, and Mauro Di Nasso. Joel Moreira spoke about results in partition regularity of equations and configurations. Alexander Fish and Michael Björklund gave lectures on applications of ergodic theory and dynamical systems to additive combinatorics while Gabriel Conant and Artem Chernikov spoke about interactions between model theory, additive combinatorics, and hypergraph regularity lemmas.

On the afternoon of the first day, there was a group problem session attended by the entire workshop. The aim of this problem session was to compile a working list of problems that might be discussed throughout the week. Subsequent afternoons were spent breaking into smaller groups to work on these and other problems. In what follows, we give brief reports of the work done in these group sessions.

Group reports

Szemerédi's theorem.

The group worked on trying to understand Szemerédi's original proof of his famous 1975 theorem on arithmetic progressions (namely that any subset of the integers of positive upper density contains arbitrarily long finite arithmetic progressions) and seeing if there was any way if nonstandard analysis could be used to simplify it. We managed to obtain (and write up) a reasonably transparent proof of the first nontrivial case of Szemerédi's theorem (Roth's theorem), and probably will be able to do the same for the general case of Szemerédi's theorem. However, thus far, nonstandard analysis has only been able to provide some very modest simplifications.

Erdő.
s' conjecture

Erdős conjectured that a subset of the natural numbers of positive lower density must contain the sum of two infinite sets. A few years ago, progress was made by Di Nasso, Goldbring, Jin, Leth, Lupini, and Mahlburg by showing that any set with upper Banach density larger than 1/2 satisfies the Erdős conjecture. The purpose of this group was to see if any further progress could be made on this conjecture.

Since no element of the group was fairly comfortable with nonstandard analysis, the group started by translating the proof in the DGJLLM paper into a standard proof. More precisely, they isolated the following lemma as the main step from the paper:

Lemma 0.1. *Let $A \subseteq \mathbb{N}$ and suppose that there are an invariant mean λ, $L \subseteq \mathbb{N}$, and $\epsilon > 0$ such that, for every finite $F \subset L$,*

$$\bigcap_{\ell \in F}(A-\ell) \cap \{m : \lambda(A-m \cap L) > \epsilon\} \text{ is infinite.}$$

Then A satisfies the conclusion of the Erdős conjecture.

From this (and a lemma of Bergelson), it follows easily that the Erdős conjecture is true for sets with upper Banach density larger than 1/2 and also for the so called pseudo-random sets in DGJLLM. The group then showed that any piecewise Bohr set also satisfies the conditions in this lemma and tried to show that any set with positive Banach density does.

The group hoped to be able to decompose any set into piecewise Bohr and a pseudo-random components and use that decomposition to prove that any set with positive Banach density satisfies the conditions in the lemma. However, they were unable to follow this strategy. The difficulty seemed to lie not so much in the decomposition but the fact that the proof that Bohr sets satisfy the conditions in the lemma is not robust with respect to to small perturbations. For instance, they could not solve the following problem:

Question 0.2. *Fix $A \subseteq \mathbb{N}$ and suppose that, for every $\epsilon > 0$, there exists a Bohr set D such that $\mathrm{BD}(D \setminus A) < \epsilon$. Does A contains $B + C$ for infinite sets B and C?*

Erdő.

s' conjecture in the model-theoretic context

The group worked on the following question: Suppose that G is an amenable group, $A \subseteq G$ is definable in an NIP expansion of G, and $\mathrm{BD}(A) > 0$. Are there infinite $B, C \subseteq G$ such that $B \cdot C \subseteq A$? Here, NIP is a model-theoretic tameness condition. This is a special case of the more general question in which one does not assume NIP, which itself is motivated by Erdős' sumset conjecture mentioned in the previous group's report. The motivation for adding a model theoretic assumption (such as NIP) comes from work of Andrews, Conant, and Goldbring which gives a positive answer to the above question in the case that G is countable and NIP is strengthened to either "stable" or "distal with elimination of $\exists^\infty$" (in the distal case, one finds that B and C are themselves definable). These statements follow quickly from the DGJLLM results mentioned in the previous group's report (for countable amenable groups) as well as work of Chernikov and Starchenko on distal structures.

During the workshop, the group used existing model theoretic results to observe the following: Let G be a definably amenable (expansion of a) group. Suppose that $A \subseteq G$ is definable and there is a left invariant Keisler measure μ such that $\mu(A) > 0$ (an assumption that follows from positive Banach density).

- If G is stable, then A is syndetic (and hence satisfies the Erdős conjecture by Hindman's theorem).

- If G is NIP, then A is piecewise syndetic.

Using this, the group removed the countability assumption from the result above on stable groups and gave a positive answer to the main question above in the case that G is abelian. It follows that the remaining difficulty is to remove the assumption that G is abelian, possibly by obtaining stronger results on piecewise syndetic sets in (non-abelian) amenable groups with NIP. The group also analyzed how to weaken the global model theoretic assumption on G to a local one about the definable set A. In the NIP setting, it is enough to assume that the family of left translates of A has finite VC-dimension. In the stable setting, it should be enough to assume that the formula $\varphi(x; y, u, v) :=$ "$yx \in uAv$" is stable. Finally, the group noted a counterexample to the second bulleted statement when NIP is replaced by "simple" (in fact supersimple of SU-rank 1). Motivated by this counterexample, the group started investigating notions of largeness for groups in simple theories.

Tangentially related to the topic at hand, it was observed that a stable subset of the integers that contains arbitrarily long arithmetic progressions is syndetic, answering a question of Gabriel Conant.

Partition regularity of equations.

Let $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial. We say that the equation $P(x_1, \ldots, x_n) = 0$ is partition regular on $\mathbb{N}$ (abbreviated as PR from now on) if one can find a monochromatic solution in every finite coloring of $\mathbb{N}$ (precisely, if for every $k \in \mathbb{N}$, for every $f : \mathbb{N} \rightarrow \{1, \ldots k\}$ there exists $a_1, \ldots, a_n \in \mathbb{N}$ such that $f(a_1) = \cdots = f(a_n)$ and $P(a_1, \ldots, a_n) = 0$). Equivalently, the equation $P(x_1, \ldots, x_n) = 0$ is PR if and only if there exists an ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ such that for every set $A \in \mathcal{U}$ there are $a_1, \ldots, a_n \in A$ with $P(a_1, \ldots, a_n) = 0$. In this case, we say that $\mathcal{U}$ is a witness of the PR of $P(x_1, \ldots, x_n) = 0$ (notation: $\mathcal{U} \models P(x_1, \ldots, x_n) = 0$).

A complete characterization of linear PR equations was given by Richard Rado almost a century ago, in terms of a very simple algebraic property of the set of coefficients:

$$\sum_{i=1}^{n} c_i x_i \text{ is PR} \Leftrightarrow \exists \emptyset \neq I \subseteq \{1, \ldots n\} \text{ s.t. } \sum_{i \in I} c_i = 0.$$

However, very little is known in the nonlinear case. The focus of this group was to produce new ideas on how to study this case, at least in some particular examples. The group did not discover any new exciting result (yet), but several rather interesting ideas and possible directions of research were proposed:

(1) Given the equation $P(x_1, \ldots, x_n) = 0$, is it possible to characterize the set

$$W_P := \{\mathcal{U} \in \beta\mathbb{N} \mid \mathcal{U} \models P(x_1, \ldots, x_n) = 0\}?$$

We know that when the equation is homogeneous, W_P is a closed bilateral ideal in $(\beta\mathbb{N}, \odot)$. In the general case, this characterization is much more difficult. The group was able to prove that, for every polynomial $P(x_1, \ldots, x_n)$, W_P does not contain selective ultrafilters. This led to ask for a characterization of these sets in terms of the Rudin-Keisler order (if any such characterization is possible).

(2) There is a connection between partition regular configurations and partition regular equations. The group tried to see if this connection could be used to characterize some classes of partition regular equations, but no new interesting class of such objects was discovered.

(3) A rather new technique in this area consist in translating the problem "is $P(x_1,\dots,x_n)=0$ partition regular?" in nonstandard terms.
More precisely, let us say that $\alpha,\beta\in{}^*\mathbb{N}$ are u-equivalent (notation $\alpha\sim_u\beta$) if for every $A\subseteq\mathbb{N}$ we have $\alpha\in{}^*A\Leftrightarrow\beta\in{}^*A$. It is possible to show that $P(x_1,\dots,x_n)=0$ is PR iff $\exists\alpha_1,\dots,\alpha_n\in{}^*\mathbb{N}$ such that $P(\alpha_1,\dots,\alpha_n)=0$.
This characterization leads naturally to study problems in this area by means of simple algebraic manipulations of u-equivalent points in ${}^*\mathbb{N}$.
The grop tried to find a list of properties of u-equivalence which could be used to prove or disprove the partition regularity of equations. With this list of properties the group was able to show the non-partition regularity of some simple nonlinear equations in two variables (for example, the arguments showed that the equation $ax^n+by^m=0$ is PR iff $n=m$ and $a=-b$). However, a deeper study of this list of properties is needed for more general equations.

(4) Let $\mathfrak{P}$ be the set of PR polynomials. Giving a complete characterization of $\mathfrak{P}$ is currently out of reach. Two questions that seems simpler to ask (but which we have not been able to completely solve) are:
   (a) Is $\mathfrak{P}$ closed with respect to some arithmetical operations?
   (b) Is it possible to say something about the complexity of $\mathfrak{P}$ from the point of view of computability theory?

$(A+B)\cdot(C+D)$.

The basic problem that this group studies was: What kind of extra structure (beyond piecewise Bohr structure) can be found in the set $(A+B)\cdot(C+D)$ for A,B,C,D sets of positive Banach density in the integers? Sine it is known that, for any two sets of positive density A and B in the integers, the set $A+B$ is piecewise Bohr (Jin?s sumset phenomenon proved by Bergelson-Furstenberg-Weiss), the above question is completely equivalent to the following: What kind of extra structure can be found in $B\cdot D$, where B and D are both piecewise Bohr sets?

The suggestion was for the group to try to prove the following: For any two piecewise Bohr sets B and D, the set $B\cdot D$ contains a piecewise periodic set. This question is still open even for product of two Bohr sets which are not neighbourhoods of zero in the Bohr topology. The only positive result is that if B and D are Bohr neighbourhoods of zero, then the set $B\cdot D$ contains a non-trivial subgroup of the integers (a result due to Alex Fish, one of the group members). Another trivial case is when one of the two sets is thick.

The group decided that it would be instructive to analyze first the case where B and D are two Bohr sets which are not neighbourhoods of zero.

Boundary amenable groups.

A (discrete) group Γ is said to be *boundary amenable* if there is a compact space X, an action of Γ on X, and a sequence of continuous maps $x\mapsto\mu_n^x : X\to\mathrm{Prob}(\Gamma)\subseteq\ell^1(\Gamma)$ such that, for all $\gamma\in\Gamma$, one has $\lim_{n\to\infty}\sup_{x\in X}\|\gamma\cdot\mu_n^x-\mu_n^{\gamma\cdot x}\|_1=0$. Amenable groups are boundary amenable (take X to be a one-point space) while the noncommutative free groups $\mathbb{F}_r$ are the prototypical examples of boundary amenable, nonamenable groups.

Since many basic classes of groups are not known to be boundary amenable or not, we figured it might help to get a nonstandard description of boundary amenable groups.

By universality of the Stone-Cech compactification, if Γ is boundary amenable, then one can always take $X = \beta\Gamma$. Since $\beta\Gamma$ is a quotient of the nonstandard extension ${}^*\Gamma$ of Γ, the group's idea was to try to characterize boundary amenability of Γ in terms of a simple algebraic object in terms of Γ and ${}^*\Gamma$. They believe that they made good progress on such a characterization during the workshop and plan to develop this thought further.

Concluding remarks

The organizers believe that the workshop was an astounding success. The interactions between the various groups proved to be very useful and it seems apparent that there will be several directions that members of the workshop will continue to pursue in the future.

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