PDE methods in complex geometry

organized by
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Workshop Summary

Introduction

This is a brief report on the AIM workshop 'PDE methods in complex geometry' from August 26 to August 30, 2024. The workshop is devoted to exploring the geometric and analytic regularity for fully nonlinear partial differential equations (PDEs) arising from geometry and physics. Yau's solution to the Calabi conjecture initiated extensive studies on geometric PDEs including the complex Monge-Ampère equations arising from both the geometric and analytic perspectives. These equations have profound applications across many areas of mathematics such as differential geometry, complex analysis, metric geometry and algebraic geometry. The goal of the workshop is to bring together experts in these fields to explore open problems and to develop new techniques leading to solutions to some of the long-standing open questions. The main topics of the workshop include:

(1) Exploration of the geometry of Kahler metrics on singular complex spaces with or without curvature assumptions.

(2) Analyzing the connection among stability, sub-solutions and the existence of solutions for a specific class of fully nonlinear PDEs.

(3) Holder continuity of solutions for fully nonlinear PDEs, particularly those of the Hessian type. Geometric and analytic estimates for singular metrics on complex varieties.

The workshop has been an extremely productive and fruitful event for the participants and their research groups. There are quite a few projects initiated and developed during the week of the workshop. Partial results have already been established. We expect several preprints to be completed by the summer of 2025. We would like to thank AIM and its wonderful staff and leadership for all the support and hospitality. We cannot emphasize more how successful this workshop is!

The following are the research reports from the four project groups.

Group project: Holder continuity for degenerate complex Monge-Ampere equations

Group members: Bin Guo, Slawomir Kolodziej, Jian Song, Jacob Sturm

Group contact: Jian Song (jiansong@math.rutgers.edu)

Project summary: We aim to develop a geometric approach to study regularity of solutions to degenerate complex Monge-Ampere equations. Let X be an n-dimensional Kahler manifold equipped with a smooth Kahler metric θ and a smooth volume form Ω. One can consider the following complex Monge-Ampere equation

$$(\theta + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{-f}\Omega \tag{0.1}$$

with the standard normalizing condition $\int_X e^{-f}\Omega = \int_X \theta^n$. Yau's solution to the Calabi conjecture guarantees a unique smooth solution φ to equation (0.1) (up to translation) as long as $f \in C^\infty(X)$. Kolodziej establishes the uniform L^∞-estimate for φ if $e^{-f} \in L^p(X)$ for some $p > 1$. He further proves the C^α-estimate for φ based on the stability theorem for (0.1) and the regularization techniques. However, the Holder continuity (or even log continuity) for the solution φ is completely open if X is not smooth. We are developing a new approach for the Holder regularity for equation (0.1) by utilizing recent developments in complex Riemannian geometry. To achieve this, we will impose the following assumptions on X and f.

(1) X is a normal projective variety with log terminal singularities. In particular, X is $\mathbb{Q}$-Gorenstein.

(2) $f \in PSH(X, \lambda\theta)$ for some $A \geq 0$.

(3) The Ricci curvature $\omega = \theta + \sqrt{-1}\partial\overline{\partial}\varphi$ is bounded below as currents.

We should be able to complete our project by the summer of 2025 and we expect to have follow-up works on the geometric consequences of this analytic regularity result.

Group project: Local α-invariant on noncompact Kähler manifold

Group members: Bin Guo, Slawomir Kolodziej, Gabor Szekelyhidi

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Project summary: It is well-known that sup-zero normalized ω-plurisubharmonic (PSH) functions on a compact (without boundary) Kähler manifold (X, ω) enjoy a uniform exponential estimate. It remains an open question whether analogous results hold on a Kähler manifold *with boundary*. This has been proved by Kolodziej on pseudoconvex domains Ω when the metric $\omega = 0$ for PSH functions with vanishing boundary values and Monge-Ampère mass 1. The group tried to generalize the argument of Kolodziej to the general setting. Namely, assume that a Kähler manifold (Ω, ω) is an open subset of a compact Kähler manifold (X, ω') where $\omega = \omega'|_\Omega$. For any $\varphi \in PSH(\Omega, \omega)$ with $\varphi|_{\partial\Omega} = 0$ and $\int_\Omega (\sqrt{-1}\partial\overline{\partial}\varphi)^n \leq c_0$, we consider the extremal function $\tilde{\varphi} = \sup\{u \in PSH(X, \omega') : u \leq \varphi \text{ in } \Omega\}$. We can show that φ is bounded from above by maximum principle. If the contact set $\Gamma \subset \overline{\Omega}$ satisfies $\Gamma \cap \partial\Omega \neq \emptyset$, then the α-invariant estimate for the Kähler manifold (X, ω') would yield the desired estimate for φ. If $\Gamma \cap \partial\Omega = \emptyset$, one can show by comparison principle this cannot occur if c_0 is sufficiently small. However, we were stuck here if c_0 is large, and the presence of ω made the usual rescaling argument fail in this case.

Another problem we discussed is the L^1_{loc} integrability of an ω-PSH function u on a noncompact complete Kähler manifold (X, ω), satisfying the MA equation $(\omega + \sqrt{-1}\partial\overline{\partial}u)^n = e^f\omega^n$ with $\int_X (e^f - 1)\omega^n = 0$. We observed that in general there is no local L^1 bound for the function u even though one assumes both u and f have compact supports. Kolodziej constructed a bounded solution in case $n = 1$, but this may not work for higher dimensions.

We believe this problem requires certain decay conditions on ω and f, and is closely related to a (weighted) Sobolev inequality, which is unknown at this moment.

Group project: Uniqueness Theorem for complete Calabi-Yau manifold of Calabi type

Group members: Yifan Chen, Shuang Liang, Ioana Suvaina, Chuwen Wang et al

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Project summary: On the disc bundle $\mathcal{C}$ around the zero section in an ample bundle L of a compact Calabi-Yau manifold D, we have the classic Calabi ansatz which gives us a Calabi-Yau metric on a space with $\mathbb{C}^*$ action such that the metric is S^1 invariant. This ansatz $(C, \omega_{\mathcal{C}})$ serves as a model space for Tian-Yau metric ω_{TY}, which was generalized later by Hein-Sun-Viaclovsky-Zhang and Y. Chen, which lives on the following setting: Let M be a compact Kähler manifold with a smooth anti-canonical divisor D with ample normal bundle N_D on D. On $X = M \setminus D$ we have the existence and uniqueness of complete Calabi-Yau metric in each Kähler class of X that is close to the Calabi ansatz in a polynomial rate under the fixed diffeomorphism given by the exponential map.

However, we do not know that whether there exists a Calabi-Yau metric $\tilde{\omega}$ that is close but not polynomially close to the Calabi ansatz $\omega_{\mathcal{C}}$. We want to prove the uniqueness theorem in this case.

According to the previous argument by Hörmander L^2 estimate, we know that we can always write $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}l$ with l has at most slightly faster than quadratic growth. And the goal is to prove that l has growth order smaller than $\frac{2}{n+1}$.

In the discussion, we tried to do rescaling and lift to the universal cover to transfer this question into a local PDE problem of $\det(\tilde{l}_{i\bar{j}} + \mathrm{Id}) = 1$ in the quasi-atlas with metric equivalent to a Euclidian metric. Another approach is that when $n = 2$, we can translate it into a Liouville theorem for an almost Calabi-Yau manifold, or a nonlinear ODE. However, both of the reduced question seems to be not true in general.

So on the other hand, it would be possible that we may find some other metric that is closed to this model with slower than any polynomial rate. This leads to solving Monge Ampère equation with Ricci potential decay in a very slow sense, which is the case that we don't have strong analytic tools for the general case to find a Calabi-Yau metric. But there is far more information on this specific model space that can help us to solve the Monge-Ampère equation.

Group project: Kahler-Einstein currents

Group members: Yifan Chen, Shih-Kai Chiu, Max Hallgren, Jacob Sturm, Gabor Szekelyhidi, Tat Dat To, Freid Tong et al

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Project summary: It is shown in Guedj-Guenancia-Zeriahi (2022) under certain assumptions or in particular for dimension ≤ 3. Summary for group discussion:

During AIM workshop (August, 26-30, 2024):

(1) We tried to understand the approach in the paper Guedj-Guenancia-Zeriahi (2022).
(2) Then we discussed on the approach of Gabor Szekelyhidi on his recent paper "Singular Kahler-Einstein metrics and RCD spaces"
(3) We tried to modify the Szekelyhidi's approach using approximations for singular Kahler- Einstein metrics via Kahler-Ricci flow or conical Kahler-Einstein metrics.
(4) We realized that the approach using the approximation of conical Kahler-Einstein metrics should work. We then discussed more in details on this direction.

After the AIM work: We are writing the details on our discussion during AIM workshop, and discussing on possible application or other questions related to this problem as well.