Partial differential equations and conformal geometry

organized by
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Workshop Summary

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Conformal geometry is the study of properties of spaces and mappings that depend on the measurement of angles but not of lengths. Although it is a subject with classical roots in complex analysis, it remains a vibrant subject. In particular, the unexpected connection between the work of Fefferman and Graham on constructing conformal invariants in Riemannian geometry to the AdS/CFT correspondence in quantum gravity has led to many exciting developments in differential geometry, partial differential equations, and mathematical physics. The aim of the workshop was to bring together a group of experts who have been working on different aspects of this rich interaction in order to foster further developments, and to identify new questions.

Overview

The workshop was held the week of August 29 to September 2, 2022. Each morning there were two talks with the aim of introducing participants to the ideas, goals and important problems in the various subjects. Since the meeting was hybird, half of the speakers were in-person and half were given by remote participants via Zoom. The speakers were (an asterisk indicates a remote speaker):

Monday: Robin Graham (University of Washington), Andrea Malchiodi* (Scuola Normale, Pisa)

Tuesday: Colin Guillarmou* (Ecole Normale Superior, Paris), Alice Chang (Princeton)

Wednesday: Piotr Chrusciel* (University of Vienna), Jie Qing (UCSC)

Thursday: Marc Herzlich* (Montpelier), Yi Wang (Johns Hopkins)

Friday: Fang Wang* (Shanghai Jiao Tong University), Xiaodong Wang (Michigan State)

A meeting wide discussion on Monday afternoon generated the following problems of potential mutual interest:

(1) **Robin Graham, University of Washington**. Relate the Mobius energy of an un-knotted loop γ in $\mathbb{S}^3$ to the renormalized area of a minimal disk $D \subseteq \mathbb{H}^4$ with $\partial D = \gamma$.

(2) **Stephen McKeown, University of Texas at Dallas**. Suppose we are given a 4-manifold $(M^4, [g])$ with $\partial M^4 \neq \emptyset$ and a 2-dimensional corner Σ. Does there exist a unique $\tilde{g} = e^{2w} g$ such that

$$\begin{cases} Q_{\tilde{g}} &= 0 \text{ in } M^4, \\ T_{\tilde{g}} &= 0 \text{ on } \partial M^4, \\ U_{\tilde{g}} &= \lambda \text{ on } \Sigma. \end{cases}$$

Remarks.
(a) Notes: 1) λ is determined by Chern-Gauss-Bonnet.
(b) Model: $M^4 = B^4_+$, $g = g_{eucl}$, $\Sigma = S^2$.
(c) Here $U_{\tilde{g}}$: leading part is Gaussian curvature, various combinations of mean curvature...

(3) **Cheikh Ndiaye, Howard University**. Suppose we are given a closed 4-manifold $(M^4, [g])$ which is not conformally equivalent to the standard 4-sphere. and a conformal class for which $P_g \geq 0$ and trivial kernel, $k_g = 8\pi^2$, study the flow

$$\frac{\partial g}{\partial t} = (-P_g Q_g) g.$$

Questions:
(a) Does solution exist globally in time? Converge (to constant Q-curvature)?
(b) It is known for the subcritical case but not via flow methods.

(4) **Robin Graham, University of Washington**. Let (X^{n+1}, g_+) be a Poincare Einstein manifold and $Y^{k+1} \subseteq X^{n+1}$, k odd, a minimal submanifold which is regular at infinity. Does there exist a local conformal invariant $\mathcal{I}$ such that

$$A = c_{k,n}\chi(Y) + \int_Y \mathcal{I}\, dS?$$

Remarks
(a) e.g. $k = 3$, $n = 4$ formula of Tyrrell? (arXiv:2204.06663 [math.DG]);
(b) $k = 1$ Mazzeo-Alexakis (done) (arXiv:0802.2250 [math.DG])

(5) **Andrew Waldron, University of California - Davis**. (A) Generalize formula of G-M-T (see below) to higher dimensions.

$$\pi^2(4\chi(X_+) - \chi(\Sigma)) = \frac{3}{4}V_+ + \frac{1}{8}\int_{X_+} |W|^2 dv + \int_Y C\, dS$$

(B) Generalize (A) to higher co-dimension. Dimension of X_+ is even, dimension of Y is odd.

(6) **Jeffrey Case, Penn State University**. Consider a closed 6-manifold M^6. The space $\mathcal{S}$ of global conformal invariants is 4-dimensional: $\mathcal{S} = \text{span}\{I_1, \cdots, I_4\}$.
Theorem (Case, 2022): $I_1([g]) \geq 0$ if $[g]$ admits an einstein metric. (arXiv:2207.00645 [math.DG])

Definition

$$I_i(M^6) := \sup_{[g]} I_i([g])$$

(A) Does there exists a M^6 such that $I_1(M^6) < 0$?

(B) Does there exist a closed manifold M^6 with $\pi_1(M^6) = \infty$ (infinite fundamental group) and $c_1, c_2, c_3 \geq 0$ such that $\sum_i c_i I_i(M^6) < 0$?

(7) **Yi Wang, Johns Hopkins University**. Assume $k < \frac{n}{2}$, $n \geq 3$. Let $[g]_{k-1} = \{g_w = e^{-2w} g_{eucl} : Vol(g_w) = 1, A_w \in \Gamma_{k-1}^+\}$, where Γ_k^+ positive k-cone.

Question: is

$$\inf_{g \in [g]_{k-1}} \int \sigma_k(g_w^{-1} A_w) dv_{g_w}$$

attained? In particular, can this be proved using concentration-compactness arguments? Can minimizers be classified? (Are they bubbles?)

(8) **Jie Qing, University of California - Santa Cruz**. Are there any "curvature quantity" which transforms under conformal changes via the p-laplace operator? Note: $\int |\nabla u|^n \, dv_g$ is conformally invariant. In general, is there a conformal invariant operator whose leading term is the p-laplace operator?

(9) **Alice Chang, Princeton University**. X^{n+1} compact manifold with $\partial X^{n+1} \neq \emptyset$.

$$Y_1(X, \partial X, [g]) = \inf_{\tilde{g} \in [g]} \frac{\int_X R_{\tilde{g}} \, dv_{\tilde{g}} + c_n \int_{\partial X} H_{\tilde{g}} \, dS_{\tilde{g}}}{(Vol(X, \tilde{g}))^{\frac{n-1}{n+1}}}$$

$$Y_2(X, \partial X, [g]) = \inf_{\tilde{g} \in [g]} \frac{\int_X R_{\tilde{g}} \, dv_{\tilde{g}} + c_n \int_{\partial X} H_{\tilde{g}} \, dS_{\tilde{g}}}{Vol(\partial X, \tilde{g})^{\frac{n-1}{n}}}$$

(X^{n+1}, g_+) CCE, $[g_o]$ = conformal infinity. Assume $Y(\partial X, [g_o]) > 0$.
(Chen-Lai-Wang) $Y_2 \geq c(n) Y(\partial X, [g_o])$
(Gursky-Han) $Y_1 \geq c_1(n)$(isoperimetric ratio) $\cdot \, Y(\partial X, [g_o])$
(Chang-Ge) $Y_1 \geq d_n Y(\partial X, [g_o])$ (no dependence on isop. constant)
What is the best d_n? Is it attained? (If so, characterize)

(10) **Alice Chang, Princeton University**. Suppose $(\mathbb{R}_+^{n+1}, g_+)$ satisfies: i) $Ric(g_+) = -n g_+$, ii) $y^2 g_+$ extends to $\mathbb{R}^n \subset \mathbb{R}_+^{n+1}$ such that (iii) $y^2 g_+|_{R^n} = e^{2f} g_{eucl}$. Under what condition(s) does it follow that $g_+ = g_{hyp}$?

Remark Some suggestions of possible conditions:

(a) Paul Yang: perhaps a suitable condition is to assume $\int_{\mathbb{R}_+^{n+1}} |W(g_+)|^{\frac{n+1}{2}} \, dv_{g_+} < \infty$.

(11) **Matthew Gursky, Notre Dame** Let g_o be a conformally compact metric defined on $\mathbb{B}^4$. Assume $Y(\mathbb{S}^3, [\rho^2 g_o]) > 0$, where $\rho > 0$ is a defining function. Consider the normalized Ricci flow

$$\begin{cases} \frac{\partial g}{\partial t} & = -2g - 6\mathrm{Ric}_g \\ g(0, \cdot) & = g_0. \end{cases}$$

Analyze singularity formation.

In the afternoons, Tuesday through Friday, the participants divided into four research groups, which worked on problems (2), (4), (10) , (11). Below we summarize progress made during (and in some cases, immediately after) the meeting:

Problem (2) Group 2 worked the problem of prescribing constant U-curvature at the corner of a compact four-manifold with corner (with vanishing Q- and T-curvatures at the interior and boundaries, respectively) via conformal change. They quickly decided to focus on the model case of the half-ball. They came up with a promising approach by conformally changing the Paneitz operator to that of hyperbolic space and separating variables. The problem was essentially reduced to computation, and ten days later, it is very near completion.

Problem (4) Work of Alice Chang, Jie Qing, and Paul Yang shows that the renormalized area of an even-dimensional Poincaré–Einstein manifold (X^{n+1}, g_+) is a linear combination of the Euler characteristic $\chi(X)$ and the integral over X of a local conformal invariant. We considered the analogous problem for submanifolds: If Y^{k+1} is an even-dimensional minimal submanifold of a Poincaré–Einstein manifold (X^{n+1}, g_+) which is regular at infinity, is it true that the renormalized area of $(Y^{k+1}, g_+|_{TY})$ is a linear combination of $\chi(Y)$ and the integral of a local conformal submanifold invariant? Prior to this workshop, the answer to this question was known to be 'yes' in the case that Y is two-dimensional; additionally, Tyrrell found a formula which almost exhibits the renormalized area in this form when Y is four-dimensional and X is five-dimensional.

During the workshop we studied this problem in three ways.

First, we affirmatively answered the above question in the case studied by Tyrrell. His formula involves the total integral of $\Delta|\mathring{\mathrm{II}}|^2 + 2|\mathring{\mathrm{II}}|^2$, which is not a local conformal hypersurface invariant. We found a local conformal hypersurface invariant which reduces to this expression under the assumptions that g_+ is Poincaré–Einstein and Y is minimal. The key was to use a local conformal hypersurface invariant, found by Blitz, Gover and Waldron, involving two derivatives of the second fundamental form. We expect that Tyrrell's derivation of his formula can be adapted to the case of higher codimension.

Second, we discussed a possible strategy for solving the general problem based on an approach published by Anastasiou, Araya, Arias, and Olea in JHEP. We determined that their argument was flawed, in that it incorrectly asserted that certain divergent integrals were in fact convergent. During the workshop we consulted with these authors and confirmed the mistake with them.

Third, we discussed the ingredients necessary to adapt the Chang–Qing–Yang argument to our problem. One ingredient is a submanifold version of Alexakis' classification of global conformal invariants. We determined that it is reasonable to conjecture that any local Riemannian submanifold invariant whose total integral is conformally invariant on closed submanifolds should be a linear combination of the Pfaffian of the intrinsic curvature, a local conformal hypersurface invariant, and the divergence of a natural vector field. Another ingredient is a suitable replacement for the Q-curvature. We discussed the relative merits of the intrinsic Q-curvature and the extrinsic Q-curvature of Gover, Waldron and collaborators, but did not settle on a "best" definition for the Chang–Qing–Yang argument.

Problem (10) This problem is a particular example of a more general question: to what extent do some of the well known fundamental results on CCE manifolds with compact closed boundary can be extended to the setting of the upper half space $\mathbb{R}^{n+1}_+ := (x, y)|x \in \mathbb{R}^n, y > 0$ with the Euclidean space $\mathbb{R}^n$ as its conformal infinity? We explored two problems in this direction:

(1) On $(X^d, g+)$ a CCE manifolds with conformal infinity (M, h), if M is compact, then $g+$ is the unique "singular Yamabe metric".

In the special case of the setting $(\mathbb{R}^{n+1}_+, g_H)$, where g_H is the standard hyperbolic metric, our group is working if there is the possibility that in addition to g_H, a second singular Yamabe metric.

We have formulated the problem into a PDE setting, and are working on the problem if the solution is upper bounded by the standard solution, then it is the standard solution.

(2) We also ask if g_H is the unique Poincare Einstein metric on $\mathbb{R}^{n+1}$ in this setting. This appears to be a much more difficult problem. Only in the case when the Einstein metric $g+$ has Weyl curvature vanishing and is complete, we know $g+ = g_H$. We propose to further study the problem under the assumption that its Weyl curvature be in $L^{(n+1)/2}$.

Six people in the group are interested in continuing the study of problems above, in particular problem (1). We have been in frequent contact since the meeting.

Problem (11) A normalized Ricci flow on X^{n+1} with an initial complete metric g_0 is a family of metrics $g(t)$ satisfying

$$\begin{cases} \partial_t g = -2(Ric + ng) \\ g(0) = g_0 \end{cases}$$

The short time well-posedness is equivalent to that of the Ricci flow, which has been established by W. Shi when the initial metic is of bounded curvature. It is known that the normalized Ricci flow preserves the conformal infinity for appropriate initial metrics.

The stability under the normalized Ricci flow for appropriate perturbations of hyperbolic spaces or non-degenerate asymptotically hyperbolic Einstein manifolds in general have been studied extensively. In fact, the normalized Ricci flow has also been used to recover perturbational existence results of Graham-Lee and Lee.

To study long time behavior of the Ricci flow is incredibly challenging and out of reach in higher dimensions. And it is expected to be so, if not more, for asymptotically hyperbolic initial metrics. Particularly, one faces the additional difficulty to control the asymptotic behavior in space at any time and preferably uniformly in time. Therefore, quickly, the working group decided to look for problems that are plausible for us to tackle in lower dimensions. The group felt, before any serious consideration for the problems in dimensions 4, it seems interesting to understand the long time behavior and convergence in dimensions 2, as well as interesting and highly nontrivial to study the developments of singularities, the long time behavior, and convergence at time infinity in dimensions 3.