Nonlinear PDEs of mixed type arising in mechanics and geometry

organized by
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Workshop Summary

What would a group of mathematicians working in gas dynamics and transonic flow have in common with a group of mathematicians working on the differential geometry of surface theory and higher-dimensional manifolds? This workshop was formed in order to answer that question. And the bottom line seems to be: Quite a lot.

The workshop was devoted to exploring common themes in the two subject areas. The unifying idea is that the underlying partial differential equations (PDEs) that arise in these fields are remarkably similar.

A little background is in order. Just as conic sections in the plane can be classified into three types—ellipses, hyperbolas, and parabolas—so PDEs can be classified into elliptic, hyperbolic, and parabolic type. And the classification proceeds along similar algebraic lines. The PDEs studied by the mathematicians at this AIM workshop have the remarkable and complicated property that the *type* of the equation actually changes when a certain curve (called the *sonic line*) is crossed.

The modern origins of these ideas originates in work of Francesco Tricomi (1897-1978). He was studying the theory of aerodynamics. His work was remarkably prescient. It turns out that, when a plane breaks the sound barrier there are regions around the plane exhibiting subsonic flow and there are other regions exhibiting supersonic flow. These two regions are separated by an unknown "free boundary". The gases may go supersonic even when the airplane is traveling at subsonic speed.

The mathematical model for the situation described in the last paragraph is a PDE that has regions of ellipticity and hyperbolicity (corresponding to subsonic and supersonic) separated by a sonic line. Tricomi was able to linearize these equations about the sonic line; the linear equation that he obtained is now known as *Tricomi's equation.*

A simple example of the type of equation we are considering here is

$$\left[\frac{\partial^2}{\partial x^2} + x \cdot \frac{\partial^2}{\partial y^2}\right] u = f$$

in the plane $\mathbb{R}^2$. Note that when $x > 0$ then the equation is (a variant of) the Laplace equation from mathematical physics. When $x < 0$ then the equation is hyperbolic. The line $x = 0$ is of course the sonic line.

We say that an abstract surface is a *Riemannian manifold* if it is equipped with a natural notion of distance. Many such surfaces, like the hyperbolic plane, are not presented as embedded in space, but rather are defined from purely mathematical considerations. Nonetheless, we would like to embed the surface into a Euclidean space so that the distance structure is preserved. This is called an *isometric embedding.* The problem of isometrically embedding a 2-dimensional Riemannian manifold into 3-dimensional space $\mathbb{R}^3$ turns out to be an analytically tight situation (there are no degrees of freedom built into the problem) that is governed by a partial differential equations of mixed type such as we have been discussing. [The problem of isometrically embedding a surface into *higher* dimensional space was solved by John Nash in the 1950s. Nash went on to win the Nobel Prize in Economics.]

The underlying PDE in this last problem is called the Gauss-Codazzi equation. It turns out that the region corresponding to positive curvature of the surface corresponds to ellipticity of the PDE, while the region corresponding to negative curvature corresponds to hyperbolicity of the PDE. The sonic line is the place where the curvature is equal to 0.

In the classical theory of elliptic PDEs, it the highest-order terms that govern the behavior of the equation and its solution. Lower order terms turn out not to matter very much. But in the study of the Gauss-Codazzi equation and the other PDEs considered here, lower order terms are very important; they lend subtlety to the situation.

An exciting new feature in this subject area is that change of type issues are a key element of string theory. This is a hot area in mathematical physics today. Fields Medalist S.-T. Yau came to the workshop to give a special presentation on this topic.

PDEs of mixed type are a meeting ground of mathematics and physics, and also of different areas of mathematics. They illustrate the dynamic and ever-growing nature of mathematical research.

The workshop kicked off with overview presentations by T.-P. Liu on gas dynamics and R. Schoen on differential geometry and isometric embedding of two-dimensional Riemannian manifolds into three-dimensional Euclidean space. After these presentations we had a problem session led by C. M. Dafermos where participants suggested open accessible problems that could be worked on. In fact, Dafermos's recent work on nonhomogeneous systems of conservation laws (motivated by problems in mechanics) has never been applied to differential geometry and isometric embedding problems and fortunately for us Dafermos himself was able to provide in our afternoon breakout session the details which may make his program applicable in geometry. On subsequent days we split the morning session into a geometry presentation (Q. Han, J. Clelland) and a gas dynamics presentation (J. Smoller, M. Feldman) with M. Slemrod in his talk attempting to bridge the gap and emphasize the unity of the two areas. Han's talk was especially useful in that it provided a clear statement of the isometric embedding problem for embedding a general n-dimensional Riemannian manifold into n(n+1)/2-dimensional Euclidean space. Particularly importance for our audience was the issue that, for n greater than or equal 3 , the embedding problem cannot be of elliptic PDE type and hence is either hyperbolic or change of type. Afternoons were spent in break out sessions thought at times the group insisted upon remaining together and discuss common mathematical issues. For example, J. Clelland and Q. Han expanded on Han's morning talk and a led a discussion as to what the generalization of the Gauss-Codazzi equations that appear in the classical embedding of two-dimensional Riemannian manifolds in three-dimensional Euclidean space would be for the general embedding problem. M. Feldman discussed some of recent developments on the existence and regularity of global regular shock reflection-diffraction configurations that involves two kinds of transonic flow and requires a deep understanding of a nonlinear partial differential equation of mixed hyperbolic-elliptic type. On Friday as noted above we had the honor of hearing S.-T. Yau lecture on geometric issues arising in theoretical physics. However, as we hoped he led us to a problem in isometric embedding with change of type partial differential equations.

While of course we cannot speak for all participants we can make to personal observations. First is that enthusiasm displayed by participants indicates that indeed there is something for these two groups to talk to each other about. Secondly the meeting provided for some of us new possible collaborators for continued research in the interplay between mechanics and geometry. The fact that J. Clelland (a geometer) and G.-Q. Chen, M. Slemrod, D. Wang (mechanics oriented nonlinear PDE people) are considering doing an AIM SQUARES proposal could only have come about via the mechanism of this AIM workshop.