Percolation on transitive graphs

organized by
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Workshop Summary

Percolation on the lattices Z^d has been a central topic of statistical physics and probability for several decades, see [Grimm]. A systematic study of percolation on general transitive graphs (most importantly, on Cayley graphs of finitely generated infinite groups) was started much later by Benjamini and Schramm [beyond]. They proposed that the most important phenomena of percolation on groups (similarly to the better understood subject of random walks) should be determined by large-scale geometric properties of the Cayley graphs. Since then, a lot of progress have been made on different aspects of this idea [LP-book], but still, many basic conjectures have proven to be rather difficult and remained unsolved. The aim of this workshop was to bring together people working in geometric group theory and probability in order to start a deeper a communication between these fields: to generate new momentum to solve some of the persistent open problems and to have new people propose new questions.

Most of the group theory people were new to percolation, hence their principal goal was to learn the main problems and techniques developed so far, to understand the main challenges, to see how their expertise could be helpful for the subject. On the other hand, the main wish of the probabilists was to learn about the asymptotic group theory techniques that seemed to be relevant to percolation, and to see specific groups that could serve as (counter-)examples: there are general percolation results (and conjectures) concerning all amenable or all non-amenable or all finitely presented groups, but few probabilists are well-acquainted with infinite groups other than Z^d, the free groups F_d, and the lamplighter groups $F \wr Z^d$.

Bringing up these two different groups of participants to a common basis from where working together could start was harder than we had expected. To help the participants prepare for the workshop, three or four weeks beforehand we compiled an annotated reading list, but only few people had time for this preparation. A typical morning of the workshop featured one introductory talk from percolation and one from group theory. These always were extremely interactive, and thus typically lasted longer than expected, and sometimes continued in the afternoon. Especially at the beginning of the week, some of these talks were unfortunately boring for the expert halves of the audience, but people getting bored with a talk formed smaller discussion groups — it was a useful feature of the workshop space that people could drift in and out without disturbing the main activity. In any case, these "learning sessions" seemed necessary and useful for many people, and the general feedback we received at the end of the workshop was that people are very happy with how much they have learned.

The Monday talks were by Russell Lyons on the main problems and techniques of percolation on groups, and by Mark Sapir on hyperbolic groups and asymptotic cones of groups. The relevance of asymptotic cones to percolation is that they could be the spaces where the scaling limit of percolation lives; in particular, Sapir conjectures the following universality phenomenon: if two groups have isometric asymptotic cones, then they behave the same way at the critical percolation parameter p_c, e.g., they must have the same critical exponents. This idea was discussed in a session on Tuesday afternoon, where the probabilists pointed out that the natural metric for the interesting scaling limits of stochastic processes on Z^2 is always the Euclidean metric, hence there seems to be a need for a modified construction of asymptotic cones, that gives the L^2 metric canonically for the case of Euclidean lattices, independently of the generating set chosen. Itai Benjamini suggested that using uniform random geodesics could be the basis for such a construction. This idea was later tested on the Heisenberg group, in discussions led by Pierre Pansu, an expert on the geometry of the asymptotic cone of the Heisenberg group. The result is negative in the sense that this uniform random geodesic metric is still dependent on the generating set, but this idea will certainly be explored more.

A special case of Sapir's conjecture on asymptotic cones is the widely believed conjecture that all non-elementary hyperbolic groups should have mean field critical percolation behavior. In fact, most groups should exhibit mean field criticality: it is known for highly non-amenable groups, for virtually free groups, and for Z^d with large enough d. The Z^d case is proved using lace expansion, a difficult technique that was the topic of a lecture on Tuesday by Antal Járai, who explained the main ideas very clearly. Wednesday afternoon had a very successful discussion session on the possibility of using lace expansion for hyperbolic groups, at least for the self-avoiding walk model, which, among many other statistical physics models, is expected to have the same scaling limit for mean field groups as percolation. Thus there is now reasonable hope to prove Sapir's conjecture for this case; the discussions of the details later in the week included Járai, Sapir, Gábor Pete, and Indira Chatterji.

There was also an idea to relate some features of lace expansion to the rapid decay property, but after an exposition by Chatterji on the RD property, the participants of the Wednesday session decided this was a mistaken trail. Loosely related to the RD property, Andreas Thom popularized some random walk versions of the Atiyah conjecture on ℓ^2-Betti numbers, e.g.: does the return probability always satisfy $p_n(x,x) = o(\rho^n)$, where ρ is the spectral radius of the walk?

Another question on asymptotic cones was whether the scaling limit of percolation could be used to define some additional structure on the real tree that is the asymptotic cone of all hyperbolic groups. In particular, Sapir and Bálint Virág formulated the following question: if a Schramm-Loewner Evolution $SLE(6)$ loop in the hyperbolic plane is conditioned to have diameter going to infinity, does it converge to Aldous's Continuum Random Tree, embedded in the hyperbolic plane in some nice way?

Tuesday morning also featured a talk by Igor Mineyev on ℓ^∞ and bounded cohomology, and his construction of a nice new metric on hyperbolic groups. The afternoon discussion session briefly examined if this construction could be relevant to find the "best" metric for scaling limits on the asymptotic cones, but the answer seems to be negative.

A recurrent question from group theorists was what the importance of the values and computability of p_c and p_u are. Interestingly, studying the supremum of p_c over all generating sets of a given group has never been considered by probabilists before. Recent work by Iva Kozáková computes p_c for many free products, and all her values are algebraic and less than about 0.52. Is there a group with transcendental p_c? Kozáková pointed out in this discussion that her work also answers a question of Yuval Peres he had wanted to know for over ten years, the value of p_c for the "grandmother graph".

Wednesday morning, Damien Gaboriau talked about ℓ^2-cohomology, Betti numbers, cost of groups and measurable equivalence relations, and the use of these notions in percolation: 1) the survival of the existence of non-constant harmonic Dirichlet functions under percolation; 2) proving the basic conjecture $p_c < p_u$ for groups with non-zero first ℓ^2-Betti number. Briefly in the morning, and in more details in an afternoon ℓ^2-discussion session, Miklós Abért talked about the cost and rank gradient and Betti numbers of graph sequences. There was a vigorous (though so far unsuccessful) discussion led by Peres on how to prove Gaboriau's conjecture $\mathrm{cost}(G) = \beta_1^{(2)}(G)+1$ using probability: is there an ϵ-density invariant percolation for any $\epsilon > 0$ that can be added to the Free Uniform Spanning Forest to make it connected? A somewhat related development is that Ádám Timár and Todor Tsankov have come up with a plan to show the indistinguishability of the trees of the Free Minimal Spanning Forest.

Gaboriau and Mineyev started working on finding probabilistic interpretations for ℓ^p-cohomologies, when $p \neq 2$.

Thursday was devoted to the geometry of percolation clusters. Firstly, Yuval Peres described an unpublished work of Schramm, "the most beautiful proof in the field", which points toward establishing $p_c < p_u$ for non-amenable groups. One possible way to complete this line of thought would be to show that the metric distortion between the Cayley graph and its infinite percolation clusters is small in a strong sense. Secondly, Timár described his elegant proof of $p_c < 1$ and $p_u < 1$ for finitely presented groups, and more generally, for groups with at most an exponential growth of the number of minimal cutsets. It is not known whether all groups satisfy this property. Sapir on Friday described some monster group constructions that could help in finding a counterexample (if there is any). On Thursday afternoon, Gábor Pete talked about his method of proving the survival of large-scale geometric and random walk properties of a Cayley graph under percolation, which also showed some connections between the lectures of Gaboriau, Peres, and Timár. He continued on Friday, with a sketch of percolation renormalization on Z^d and describing some open questions motivated by trying to generalize this procedure to other groups.

Betti numbers were the topic of a spontaneous Friday afternoon discussion session led by Andreas Thom. There are several ℓ^2 and group theory questions (not really related to percolation) where progress seems to have been made; these developments included Thom, Abért, Chatterji, and Martin Kassabov.

Since there have been a lot of discussions also in smaller groups, often lasting till late night, there are certainly further developments that we are not aware of.

In summary, we are very happy with the workshop. Good progress has been made in a number of problems that were in the focus of the workshop. New collaborations have been established, with the active participation of several graduate students and postdocs. Some exciting new directions have emerged. The ideas and problems of percolation have infected quite a few group theorists, while probabilists gained a better understanding for what type of percolation problems the group theory techniques could be useful.

Bibliography

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