D. Pickrell AIM, Apr 9 2013

$M = RS$, K cpt, $G = K^{\mathbb{C}}$

$$A(g) = \frac{1}{2}\int \langle g^{-1}dg \wedge * g^{-1}dg \rangle$$

Harm. maps: $\partial(g^{-1}\bar\partial g) + \bar\partial(g^{-1}\partial g) = 0$ Why compl. $\int$ble?

$0 \neq \lambda \in \mathbb{C}$

$$A^\lambda = \frac{1}{2}\left(1 - \frac{1}{\lambda}\right) g^{-1}\frac{\partial g}{\partial z}dz + \frac{1}{2}(1-\lambda)g^{-1}\frac{\partial g}{\partial \bar z}d\bar z$$

$\lambda = 1$: trivial $\lambda = -1$: $g^{-1}dg$

Th. 1 g harmonic iff A^λ flat $\forall \lambda \neq 0$: loop in space of fl. conn.

Pohl. Meyer, Zakharov — —

$$dA^\lambda + \frac{1}{2}[A^\lambda \wedge A^\lambda] = 0$$

Harm. map $\Longleftrightarrow$ $*(g^{-1}\partial g)$ is flat

0 curvature repn.

Uhlenbeck / G. Segal:

For simplicity $M = S^2$

Def (Uhlenbeck) An extended soln is a map $E(z, \lambda)$: $M \times (\mathbb{C}\setminus\{0\}) \to G$

s.t. 1) holo in λ

2) reality condition: $|\lambda| = 1 \Rightarrow E \in K$

$$E^{-1}dE = \frac{1}{2}\Big((1-\lambda^{-1})g^{-1}g_z dz + (1-\lambda)g^{-1}g_{\bar z}\, d\bar z\Big)$$

3)

$$\begin{array}{ccc} \{\text{Ext solns}\} & & E(z,\lambda) \\ \downarrow & & \downarrow \\ \mathrm{Harm}(M,K) & & g(z) = E(z, \lambda = -1) \end{array}$$

$$\begin{array}{ccc} \{k \in LK\} & \longrightarrow & \text{Ext solns} \\ \wr & & \downarrow \\ k : \mathbb{C}^* \xrightarrow{\text{holo}} G : & & \text{Harm} \end{array}$$

$|\lambda| = 1 \Rightarrow k(\lambda) \in K$

$$\begin{pmatrix} a & bz \\ -\bar{b} z^{-1} & \bar{a} \end{pmatrix}$$

$z \longmapsto E(z,\lambda)$

Ext. soln: $E : M \to \Omega K = \{ h : \mathbb{C}^* \xrightarrow{\text{holo}} G \mid h(S^1) \subseteq K,\ h(1) = 1_K \}$

Harm. maps $g : M \longrightarrow K$

(Segal)

Pressley–Segal: ΩK is a cx mfd

cf $K/T \cong G/B^+$

$LK/K \cong \Omega K \cong LG/L^+G$ holo in disc

Factoris^n result: Gram – Schmidt

$$g \in LG \Rightarrow g(\lambda) \overset{!}{=} \underset{\substack{\Uparrow \\ \Omega K}}{k(\lambda)}\, \underset{\substack{\Uparrow \\ L^+G}}{g_+(\lambda)}$$

g_+ holo in $\Delta = \{|\lambda| < 1\}$

Action of LG on space of harm. maps

Bäcklund xfns

—

$LG \times \Omega K \circlearrowleft$:

$\big(h(\lambda) \cdot k(\lambda)\big)_{\Omega K}$

factoriz^n unobstructed: cf. $\overset{G}{\overset{\shortparallel}{KAN}}$, Iwasawa

— Factoriz^n depends ctsly on ...

Th $\exists$ action

$LG \times \{\text{extended solns}\} \circlearrowleft$

$h(\lambda),\ E(z,\lambda) \to \big(z \to hE(z,\cdot)\big)_{\Omega K}$

$\mathbb{C}^*_{<1}$ acts (Morse fn is energy fn on ΩK)

$$\begin{tikzcd}
M \arrow[r, bend left, "g"] \arrow[d, bend right, "E"', "\text{holo.}"] & K \\
\Omega K \arrow[ur, "\text{eval at } -1"'] &
\end{tikzcd}$$

Twistor constr. applied to harm. maps.

Choose ext soln to go into Schubert vty

Bott–Samelson: A Schubert vty is iterated sphere bdle

Poisson Lie gp str for LG (4)

Yangian (?) — qm gp analogue

Semenov — T — S late 80s

If eval at $\lambda \neq -1$ get soln to WZW

WZW action: $\frac{1}{T} E(g) + 2\pi i l \, WZW(g)$ critical $\frac{1}{T} \sim l$

$$\left(\frac{1}{T} + l\right) \partial \left(g^{-1} \bar{\partial} g\right) + \left(\frac{1}{T} - l\right) \bar{\partial} \left(g^{-1} \partial g\right)$$