Polya-Schur-Lax problems: hyperbolicity and stability preservers

organized by
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Workshop Summary

Background and Goals

The workshop *Pólya-Schur-Lax Problems: Hyperbolicity and Stability Preservers*, held at the American Institute of Mathematics Research Conference Center on May 28–June 1, 2007, was devoted to the recent developments in the theory of distribution of zeros of polynomials and transcendental entire functions and some of their most important applications. The activities focused mainly on the following topics and their interplay:

1. Pólya-Schur problems: classification of linear preservers of polynomials and transcendental entire functions in one or several variables with prescribed zero sets.
2. Lax-type problems: determinantal representations of multivariate Gårding-hyperbolic polynomials and related objects.
3. Properties and applications of stable polynomials and polynomials with the half-plane property in combinatorics, matrix theory, optimization and statistical mechanics.

The first topic goes back to Laguerre and Pólya-Schur and is intimately connected with the other two, as shown by the recent solutions to the Pólya-Schur problem for univariate hyperbolic (i.e., real-rooted) polynomials [BBS2] and the 1958 Lax conjecture for Gårding-hyperbolic polynomials in three variables [HV,LPR], respectively. The notion of hyperbolicity in one variable has several important extensions to higher dimensions [BBS1], in particular the well-known concept of Gårding-hyperbolic polynomial [Gar]. The latter was initially studied in the theory of partial differential equations [ABG,Gar,Lax] and is closely related to other fundamental classes of polynomials, such as stable polynomials and polynomials with the half-plane property [BBS1,COSW,Fis,Gu]. Pólya-Schur problems for these classes of polynomials or variations thereof have been actively studied over the last century and have significant applications in the theory of entire functions [BE,Car,CPP,CC20,CCS21,CD22,Le], statistical mechanics [HL,LY,LS], control/stability theory and convex optimization [BGLS,HV], matrix theory, probability theory and combinatorics [J-con,Br9,Br10,COSW,Pit,St,W].

The combination of the two recent breakthroughs mentioned above – namely the solution of the Pólya-Schur problem for circular domains [BBS1,BBS2] and the proof of the Lax conjecture [HV,LPR] – has already proved to be very fruitful and is expected to have many more far-reaching consequences as we are yet to understand and fully explore it. Indeed, the notion of *real stable polynomial* – which is the central ingredient used in e.g. [BBS1,BBS2,J-con] – has became utterly important in several apparently unrelated areas [COSW] and seems to provide an appropriate framework for studying in a unified manner concepts such as Hurwitz-Schur stability, the half-plane property and hyperbolicity. The power of the notion of real stability and its further potential are also illustrated by its recent applications to the solution of several well-known conjectures in linear algebra [J-con]. Moreover, three major conjectures – namely Lieb's "permanent–on–top" (POT) conjecture in matrix theory/combinatorics, the Bessis-Moussa-Villani (BMV) conjecture in quantum statistical mechanics and the so-called "Big Conjecture" in negative dependence/probability theory – can all be reformulated and generalized in a natural way by means of real stable polynomials or closely related objects, see §–. The classification of real stability preservers obtained in [BBS1,BBS2] combined with Lax-type results [HV,LPR] lays ground for even more exciting developments as it provides new tools for dealing with yet another fundamental problem, namely the characterization of Fourier transforms with all real zeros, see § and §. This is a question of central interest in entire function theory but progress has been relatively scarce ever since Pólya's initial results and de Bruijn's subsequent work [dB,Car,CD22]. It is therefore quite clear that one must now develop a general theory of real stable polynomials that would put them on equal footing with e.g. totally positive matrices.

The purpose of this workshop was to bring together experts in the above areas who have a specific interest in Pólya-Schur-Lax problems, inform them about substantial new developments and establish a collaborative network around these subjects. Our goals were threefold:

(i) To organize a short lecture series devoted to the recent progress on Pólya-Schur-Lax problems and their main applications.
(ii) To organize discussion sessions on the most relevant projects to be pursued in these areas.
(iii) To start interactive collaboration between people from the diverse areas of research impacted by Pólya-Schur-Lax problems.

Workshop Activities and Progress Report

In sharp contrast to the usual format of conferences, this workshop provided a unique experience of interaction with researchers in diverse fields. The structure of the workshop was in the tradition crystallized at AIM with about two introductory talks each morning followed by problem sessions and group discussions in the afternoon. This format was very well-suited to our goals and needs, encouraging active involvement of all the participants and fostering new collaborations. Since the main focus of the workshop was on concrete (Pólya-Schur-Lax) *problems* (see § and §) we kept a great deal of flexibility throughout the meeting so as to give participants as much time as possible to discuss and work jointly on these problems.

The talks given over the five days of the workshop served two purposes: on the one hand, they gave a state-of-the-art report on the topics dealt with and thus became a common platform that participants from various research areas could build on; on the other hand, these talks were a lead-in and a general "warm-up" for the problem sessions and group discussions that were subsequently scheduled following suggestions from all the participants. Unlike traditional lectures, all the talks given at the workshop had an interactive nature and allowed for numerous questions and remarks from the audience. Below is a summary of these talks:

- The opening talk given by G. Csordas reviewed the history and applications of Pólya-Schur problems in the distribution of zeros of polynomials and transcendental entire functions, from the fundamental results of Laguerre, Hermite, Pólya and Schur to the most recent achievements. One of the highlights of this talk was a most intriguing reformulation of the Riemann Hypothesis (due to the speaker) in terms of multiplier sequences and CZDS (see Problem 21 in §).

- V. Vinnikov gave an overview of the Lax Conjecture and its recent proof [HV,LPR] and a survey of related problems dealing with determinantal representations for hyperbolic and real stable polynomials (see §).
- L. Gurvits presented his recent proofs and generalizations of the Van der Waerden Conjecture, Schrijver-Valiant Conjecture, Alexandrov-Fenchel and Brunn-Minkowski inequalities by means of hyperbolic and real stable polynomials and discussed some algorithmic aspects [Gu,Gu1]. The notion of *capacity* of a real homogeneous polynomial introduced by the speaker (which, as pointed out by C. Johnson, in the case of determinantal polynomials is the same as the notion of "equilibrant" of a matrix due to A. Hoffman) seems to play important role in many applications (see Problem 3).
- J. Borcea and P. Brändén proposed equivalent formulations and generalizations of the BMV Conjecture and the POT Conjecture, respectively, in terms of hyperbolic and real stable polynomials (see §–). Subsequently, C. Johnson gave a short but informative review of the results known so far and the matrix theoretic approaches to the BMV Conjecture (as was recently shown by Lieb-Seiringer [L-S], the BMV conjecture may be reformulated by means of positive definite matrices).
- O. Holtz talked about Newton inequalities and ongoing efforts for finding unifying principles (in the spirit of O. Taussky-Todd) for classes of matrices satisfying Hadamard-Fischer-Kotelijansky type inequalities and various stability properties. She also gave an overview of related negative dependence properties and applications to combinatorics, matroid theory and probability theory, including Mason's Conjecture and some of the conjectures proposed by Pemantle [P] and Wagner [W] (see §). Subsequently, C. Johnson suggested an approach to finding such principles by means of matrices whose minors satisfy the so-called LIA ("Leading Implies All") property.
- AIM Web Director D. Farmer gave an informative talk about zero spacings for the Ξ function, the dynamics of these zeros under differentiation and other transformations, and connections with the Riemann Hypothesis. He also proposed several related problems and conjectures, see Problem 1. (This was in fact the first time that one of the AIM Directors participated in the activities of an workshop held at AIM.) Shortly after the workshop, D. Dimitrov announced that together with V. Kostov he constructed counterexample to Conjecture 5.1.1 in [FR] (see Problem 1 (iv)).

There were also several ad-hoc short talks in which participants informed the audience about recent progress – in certain cases during the workshop itself – or posed new problems (some of which are listed in §):

- Y.-O. Kim and H. Ki reported new results on certain linear operators, the Riemann ξ-function and the de Bruijn-Newman constant Λ. In particular, they announced a proof of the inequality $\Lambda < 1/2$ (see Problem 5);
- D. Wagner proposed a conjecture which, if true, would imply that any so-called Rayleigh matroid satisfies Mason's ultra log-concavity conjecture (see Problem 6, § and [W]);
- M. Gekhtman gave a short overview of recent positivity results for (generalized) immanants of totally nonnegative matrices and formulated a positivity conjecture

for Hadamard-Fischer-Kotelijansky like expressions involving products of minors of certain totally nonnegative matrices associated with weighted graphs;

- O. Güler formulated a conjecture with important applications in dual hyperbolic programming problems, namely an inequality between the third and second order directional derivatives of the Fenchel dual of the self-concordant barrier function ("$-\log\det$") of a hyperbolic polynomial;
- D. Dimitrov posed the problem of characterizing Fourier transforms with all real zeros by means of minors of an associated upper triangular infinite Toeplitz matrix;
- D. Cardon promulgated an elegant geometric argument which shows that the coefficients of real entire functions whose zeros lie in a strip satisfy a certain concavity property (Newton's inequalities) enjoyed by univariate hyperbolic polynomials;
- J. Borcea and M. Tyaglov (a PhD student of O. Holtz) independently proposed arguments to solve a 171-year-old open problem of Gauss (made precise in [CCS21]) known as the Hawai'ian Conjecture (Problem H in §).

Group discussions focused on the topics that we just described. A compilation of the questions addressed by various groups over the five days of the meeting is given in § and § below. To briefly outline some of the issues dealt with during problem sessions we could mention:

- Multiplier sequences, CZDS and λ-sequences (cf. §) were a quite popular subject and made the object of several problem sessions. Attempts were made (at least) by T. Craven, G. Csordas, D. Dimitrov, S. Edwards, S. Fisk, O. Katkova, B. Shapiro, A. Vishnyakova, C. Johnson, P. Brändén, S. Friedland, J. Borcea, D. Cardon, A. Piotrowski, J. Lynn-Halfpap, E. Steinbart to characterize multiplier sequences whose reciprocals are positive definite sequences (Problem 22) and multiplier sequences which are CZDS (Problem 18). O. Katkova and A. Vishnyakova formulated an interesting conjecture giving a sufficient condition for rapidly decaying CZDS (Problem 4), T. Craven found counterexamples to an overly-optimistic conjecture claiming that non-decreasing CZDS are also preservers of the set of real polynomials with all zeros in a given horizontal strip (symmetric in the real axis), and several people brainstormed on the (quite difficult) problem of deciding whether rapidly decaying positive sequences are CZDS (Problem 20).
- M. Tyaglov, S. Fisk and J. Garloff worked on an intriguing conjecture on complex interlacing properties for the zeros of pairs of real polynomials obtained by regrouping terms of degrees which are multiples of 4 in the real and imaginary parts of an arbitrary Hurwitz stable polynomial. C. Johnson pointed out that a natural matrix theoretic approach to this problem would be to study possible connections with eigenvalues/singular values of symplectic matrices and quaternionic matrices.
- Fourier transforms, linear operators of the form $e^{-\lambda^2 D^2}$ with $\lambda \in \mathbb{R} \setminus \{0\}$ and related kernel representation problems were investigated by H. Ki, Y.-O. Kim, G. Csordas, D. Cardon, D. Farmer, D. Dimitrov. In particular, they tried to construct examples of linear operators preserving "zeros in a strip" by means of Hermite-Biehler polynomials/functions.
- Problem 16 and Problem 17 (on infinite Lax type determinantal representations for Laguerre-Pólya functions in two variables and multivariate nonnegative polynomials/sums of squares, respectively) were explicitly formulated and discussed by

V. Vinnikov, J. Borcea, P. Brändén and B. Shapiro. Partial results in the direction suggested by Problem 17 are contained in an ongoing joint work with A. Guterman.

- Problems pertaining to computational complexity issues for determinantal representations and hyperbolicity preservers as well as properties of homogeneous cones and hyperbolicity cones (relevant in convex optimization and semidefinite programming) were studied by V. Vinnikov, O. Güler, P. Parrilo, O. Shevchenko, S. Friedland, L. Gurvits. In particular, they tried to decide whether one can produce effective algorithms for testing hyperbolicity by reducing the problem "one variable at a time".
- The aforementioned talks by J. Borcea and M. Tyaglov on the Hawai'ian Conjecture (Problem H in §) generated considerable interest and led to invigorating discussions about the nature of certain level curves (called "gardens" or "sunsets" in the literature). These curves were studied by (at least) G. Csordas, T. Craven, S. Edwards, B. Shapiro, P. Brändén, J. Borcea. Based on her joint work with A. Hinkkanen, S. Edwards produced a numerical counterexample to a stronger conjecture made earlier by J. Borcea and B. Shapiro in [**?**, Conjecture 2]. The latter two participants suggested possible complex and multivariate versions of the Hawai'ian Conjecture, while B. Sturmfels (who attended the workshop for one day together with two of his PhD students at UC Berkeley) asked for possible connections with hives and "tropical" versions of this conjecture.
- O. Holtz, M. Gekhtman, C. Johnson, P. Brändén and J. Borcea worked on the capacity problem posed by L. Gurvits in his talk (Problem 3) and managed to solve it in some special cases (however, they were subsequently told by L. Gurvits that these special cases were already known to him).
- The problem posed by D. Wagner (Problem 6) made the object of a group discussion involving D. Wagner, P. Parrilo, J. Borcea and P. Brändén. They showed that the problem actually reduces to a certain non-convex optimization problem. Unfortunately, existing computer codes of P. Parrilo are not applicable due to the non-convexity of the feasibility region.
- The POT Conjecture, Schur's determinantal inequality and their multivariate real stable extensions proposed in § were discussed by M. Gekhtman, C. Johnson, O. Holtz, P. Brändén, S. Fisk, S. Friedland, J. Borcea. This led to some interesting observations concerning combinatorial properties of homogeneous real stable symmetric polynomials. P. Brändén and S. Fisk suggested possible extensions of Stanley's univariate hyperbolicity criterion (in terms of symmetric/Schur functions) to real stability in two variables.
- The BMV Conjecture was the theme of several problem sessions involving (at least) G. Csordas, S. Friedland, M. Gekhtman, P. Brändén, C. Johnson, O. Holtz, J. Borcea. The real stable and hyperbolic versions of these conjecture (proposed in §) as well as various matrix theoretic approaches were discussed. In particular, (almost successful) attempts were made by P. Brändén, M. Gekhtman and S. Friedland to settle the case when at least one of the matrices in Lieb-Seiringer's reformulation of this conjecture (see Problem 39) has rank at most two.

In conclusion, this was an enriching experience and we all viewed the meeting as highly successful. The workshop venue, with its generous spaces, wonderful library and computing and other facilities, created a relaxed milieu which was conducive for the free exchange of ideas. Thanks to the assistance and the expert guidance of the Directors of AIM, the transition from a few, brief introductory talks led seamlessly to small (and large) group discussions dealing with problems of common interest. All the participants benefited from these incipient yet dynamic collaborations that will certainly bear fruit in the near future. Indeed, we are quite hopeful that these collaborations will continue and eventually lead to significant developments on some of the problems discussed at this workshop. We would therefore encourage AIM to consider holding a follow-up meeting in a year or so, where people could report on the progress originating from this workshop and focus on new important challenges.

Problems Proposed by the Participants

Problem 1 (D. Farmer)**.** Let Φ and Ξ be as in e.g. Problem 5 below.

(i) What are the number theoretic aspects "hidden" in the integral expression

$$\int_0^\infty \Phi(x)\cos(zx)dx\,?$$

For instance, is it possible to prove the Prime Number Theorem (which is equivalent to saying that the aforementioned function in z has no zeros in the half-plane $\{z \in \mathbb{C} : \mathfrak{Im}(z) \geq 1/2\}$) using only this integral expression?

(ii) Which multiplier sequences even out zero spacings for the Ξ function?

(iii) Does there exist $R > 0$ such that the function $\int_0^R \Phi(x)\cos(zx)dx$ has more than one pair of (complex) correlated zeros?

(iv) Is differentiation a better way than the midpoint method to even out spacings of zeros of real entire functions of order 1 with all real zeros (cf. [**?**, Conjecture 5.1.1]; see also §)?

Problem 2 (S. Fisk)**.** **(i)** is a Lax-type problem for (Hurwitz) stable polynomials while **(ii)** is a Horn-type problem for stable homogeneous polynomials.

(i) If A is a skew symmetric $n \times n$ matrix and B_1, B_2 are positive definite then $\det(A + i(xB_1 + yB_2))$ is a stable polynomial, and all terms have even degree if n is even, and odd degree if n is odd. Is the converse true?

(ii) Recall the additive Horn problem:

> Fix an integer n, and let $\alpha, \beta, \gamma \in \mathbb{R}^n$. Is there a triple (A, B, C) of Hermitian matrices with $A + B + C = 0$, and eigenvalues α, β, γ?

The corresponding problem for polynomials can be phrased in a similar manner:

> Fix an integer n, and let $\alpha, \beta, \gamma \in \mathbb{R}^n$. Is there a triple (f, g, h) of real rooted polynomials with roots α, β, γ and a homogeneous stable polynomial $F(x, y, z)$ such that $f = F(x, 1, 0)$, $g = F(0, x, 1)$ and $h = F(1, 0, x)$?

It is known that Horn's problem is solvable for α, β, γ iff (α, β, γ) is the boundary of a structure called a *hive*. The logs of the coefficients of the polynomial F determine a hive. However, unlike the Horn theorem, this is not a sufficient condition. What are necessary and sufficient conditions on f, g, h that guarantee the existence of F?

Problem 3 (L. Gurvits)**.** Consider the Fischer-Fock space $\mathcal{F}_k$, i.e., the Hilbert space of holomorphic functions on $\mathbb{C}^k$ with inner product

$$\langle f, g\rangle_{\mathcal{F}_k} = \sum_{\alpha\in\mathbb{N}^k} \alpha! a(\alpha)\overline{b(\alpha)},$$

where $\sum_\alpha a(\alpha)z^\alpha$ and $\sum_\alpha b(\alpha)z^\alpha$ are the Taylor expansions of f and g, respectively (see, e.g., [BBS1]). Given a real homogeneous polynomial in k variables $p(z_1, \dots, z_k)$ of degree n define its *capacity* $\mathrm{Cap}(p)$ by

$$\mathrm{Cap}(p) = \inf_{z_1>0,\dots,z_k>0} \frac{p(z_1,\dots,z_k)}{(z_1\cdots z_k)^{\frac{n}{k}}}\cdot\frac{n!}{k^n}.$$

Let p, q be any two real homogeneous polynomials in k variables of degree n. Is it true that

$$\langle p, q\rangle_{\mathcal{F}_k} \ge \mathrm{Cap}(p)\cdot\mathrm{Cap}(q)\cdot\frac{n!}{k^n}\ ?$$

Problem 4 (O. Katkova, A. Vishnyakova)**.** Let $\Gamma = \{\gamma_k\}_{k\in\mathbb{N}}$ be a positive sequence such that $\gamma_k^2 \ge 4\gamma_{k+1}\gamma_{k-1}$ for all $k\in\mathbb{N}$. Is it true that Γ is a CZDS (see Problem 18 in § for the definition of a CZDS)?

Problem 5 (H. Ki, Y.-O. Kim)**.** Let $\Xi(z)$ be the Riemann Ξ-function:

$$\Xi(z) = \int_{-\infty}^{\infty}\Phi(t)e^{izt}dt,$$

where

$$\Phi(t) = 2\sum_{n=1}^{\infty}(2n^4\pi^2e^{9t/2} - 3n^2\pi e^{5t/2})e^{-n^2\pi e^{2t}}.$$

Define $\Xi_\lambda(z)$ by

$$\Xi_\lambda(z) = \int_{-\infty}^{\infty}\Phi(t)e^{\lambda t^2}e^{izt}dt.$$

(i) Is it true that for any $\lambda < 0$ almost all zeros of $\Xi_\lambda(z)$ are real?

If true, one of the consequences of this question would be that almost all complex zeros of the Riemann zeta function are on the critical line.

(ii) Suppose f is a real entire function of order 1 and maximal type, f is even, and the zeros of f lie in the strip $\{z : |\mathfrak{Im} z| \le \Delta\}$ for some $\Delta > 0$. Is it true that for every $\lambda > 0$ all but a finite number of zeros of $e^{-\lambda D^2}f$ are real and simple?

Let ζ be the Riemann zeta-function, and define the function Ξ by

$$\Xi(t) = \frac{s(s-1)}{2}\Gamma\left(\tfrac{s}{2}\right)\pi^{-s/2}\zeta(s) \qquad (s = \tfrac{1}{2}+it).$$

Let χ_1 denote the Dirichlet character modulo 5 such that $\chi_1(2) = i$. Put

$$\alpha = \frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1},\quad \theta = \tan^{-1}\alpha,\quad f(s) = \frac{\sec\theta}{2}\left(e^{-i\theta}L(s,\chi_1) + e^{i\theta}L(s,\bar{\chi}_1)\right),$$

and define the function Φ by

$$\Phi(t) = \Gamma\left(\tfrac{s+1}{2}\right)\left(\frac{5}{\pi}\right)^{s/2} f(s) \qquad (s = \tfrac{1}{2}+it).$$

Let χ_2 denote the Dirichlet character modulo 4 such that $\chi_2(3) = -1$, and define the function H by

$$H(t) = \Gamma\left(s + \tfrac{1}{2}\right)\left(\frac{2}{\pi}\right)^s L(s - \tfrac{1}{2}, \chi_2)L(s + \tfrac{1}{2}, \chi_2) \qquad (s = \tfrac{1}{2} + it).$$

The functions Ξ, Φ and H are even real entire functions of order 1 and maximal type. The Riemann hypothesis is the conjecture that Ξ has only real zeros. It is known that the zeros of Ξ lie in the strip $\{t : |\mathfrak{Im} t| < 1/2\}$, and that Ξ has infinitely many real zeros. The function Φ has infinitely many real zeros and infinitely many non-real zeros; and there is a positive constant Δ such that the zeros of Φ lie in the strip $\{t : |\mathfrak{Im} t| \le \Delta\}$. The function H has no real zeros at all, but infinitely many non-real zeros; and the zeros lie in the strip $\{t : |\mathfrak{Im} t| < 3/2\}$. The following theorem (due to H. Ki, Y.-O. Kim and J. Lee) provides an affirmative answer to question **(ii)** for the functions Ξ, Φ and H:

Theorem 1. *For every $\lambda > 0$ all but a finite number of zeros of $e^{-\lambda D^2}\Xi$, $e^{-\lambda D^2}\Phi$ and $e^{-\lambda D^2}H$ are real and simple.*

This theorem, in particular, implies that the de Bruijn-Newman constant is less than 1/2. In our notation, the de Bruijn-Newman constant Λ is given by

$$\Lambda = 4\inf\{\lambda \in (-\infty, \infty) : e^{-\lambda D^2}\Xi \text{ has only real zeros}\}.$$

Problem 6 (D. Wagner)**.** Let $E = \{1, \dots, m\}$ and consider a multiaffine polynomial

$$Z = Z(y_1, \dots, y_m) = \sum_{S \subseteq E} \omega(S) y^S,$$

where $y^S = \prod_{e \in S} y_e$. For $e \in E$ let $Z^e := Z|_{y_e = 0}$ and $Z_e := \partial Z / \partial y_e$, so that $Z = Z^e + y_e Z_e$. For distinct $e, f \in E$ let

$$\Delta Z\{e, f\} := Z_e^f Z_f^e - Z_{ef} Z^{ef}.$$

Assume that

(1) $\omega(S) \ge 0$ for all $S \subseteq E$,
(2) $\omega(S) \le \omega(T)$ whenever $T \subseteq S \subseteq E$,
(3) $\Delta Z\{e, f\} \ge 0$ for all distinct pairs $e, f \in E$ and for all positive real values of $y_1, \dots, y_m$.

Given a permutation τ of E let

$$\tau(Z) = \tau(Z)(y_1, \dots, y_m) = Z(y_{\tau(1)}, \dots, y_{\tau(m)}).$$

For distinct $g, h \in E$ let $\tau_{gh} = (gh)$ and set $\tilde{Z} = Z + \tau_{gh}(Z)$. The question is as follows: is it true that

$$\Delta \tilde{Z}\{e, f\} \ge 0$$

for all distinct $e, f, g, h \in E$ and for all positive real values of $y_1, \dots, y_m$?

It should be noted that by the results recently established in [BBL] it is known that if the monotonicity condition (2) is dropped then the answer to the above question is negative (see also § below).

Problems, Glossary and References Proposed by the Organizers

Linear Preserver Problems and Determinantal Representations in Entire Function Theory.

A non-zero univariate polynomial with real coefficients is called *hyperbolic* (or is in the Laguerre-Pólya class [BCC,BCCG,Bor,Car,CC20]) if all its zeros are real while a univariate polynomial f with complex coefficients is called *stable* if $f(z) \neq 0$ for all $z \in \mathbb{C}$ with $\mathfrak{Im}(z) > 0$. Hence, a univariate polynomial with real coefficients is stable if and only if it is hyperbolic.

A polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is *stable* if $f(z_1, \ldots, z_n) \neq 0$ for all $(z_1, \ldots, z_n) \in \mathbb{C}^n$ with $\mathfrak{Im}(z_i) > 0$, $1 \leq i \leq n$. If in addition f has real coefficients then f is said to be *real stable* [BBS1,Le].

A homogeneous polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ is called (*Gårding*) *hyperbolic* with respect to a given vector $e \in \mathbb{R}^n$ if $f(e) \neq 0$ and for all vectors $\alpha \in \mathbb{R}^n$ the univariate polynomial $t \mapsto f(\alpha + te)$ has only real zeros [ABG,Gar,Gu,HV,LPR].

Let $\Omega \subset \mathbb{C}$ and denote by $\pi(\Omega)$ the class of all (complex or real) univariate polynomials whose zeros lie in Ω.

Problem 7. Characterize all linear operators $T : \pi(\Omega) \to \pi(\Omega) \cup \{0\}$.

Let π_n denote the vector space (over $\mathbb{R}$ or $\mathbb{C}$) of all polynomials of degree $\leq n$. For $\Omega \subset \mathbb{C}$ (where Ω is an appropriate set of interest) let $\pi_n(\Omega)$ denote the class of all polynomials of degree $\leq n$ all of whose zeros lie in Ω. The finite degree analog of Problem 7 is as follows.

Problem 8. Describe all linear operators $T : \pi_n(\Omega) \to \pi(\Omega) \cup \{0\}$ for $n \in \mathbb{N}$.

Remark 1. Problems 7–8 originate from the works of Laguerre and Pólya-Schur and were explicitly stated first by T. Craven and G. Csordas [CC20,Cso] and subsequently by J. Borcea, P. Brändén, B. Shapiro [BBS2] (see also Q. I. Rahman, G. Schmeisser [**?**, pp. 182–183]). Various special cases of Problem 7 when $\Omega = \mathbb{R}$ have been considered e.g. by A. Aleman, D. Beliaev, H. Hedenmalm [ABH] and S. Fisk [Fis]. For additional information, see [Bor]–[J-con], [Br10] and references therein.

In [BBS2] J. Borcea, P. Brändén, B. Shapiro completely solved Problems 7–8 for all closed circular domains and their boundaries and in [BBS1] they obtained multivariate extensions for all finite order linear differential operators with polynomial coefficients. We emphasize some important remaining open cases of these problems:

Problem 9. Settle Problems 7–8 in the following situations:

(a) Ω is an open circular domain,
(b) Ω is a sector or a double sector,
(c) Ω is a strip,
(d) Ω is a half-line.

Problem 10. Any linear operator T on $\mathbb{C}[z]$ may be uniquely represented as a formal linear differential operator with polynomial coefficients (cf. [BBS1]), i.e., $T = \sum_{k=0}^{\infty} Q_k(z) \frac{d^k}{dz^k}$, where $Q_k \in \mathbb{C}[z]$ for all $k \geq 0$. Characterize the polynomials Q_k such that $T : \pi_n(\Omega) \to \pi(\Omega) \cup \{0\}$ for $n \in \mathbb{N}$ (cf. Problems 7–8).

Problem 11 ([CC20)**.**] Let π_n denote the vector space over $\mathbb{R}$ of all polynomials of degree $\leq n$. Characterize all linear transformations (operators) $T : \pi_n \to \pi_n$ such that

$$Z_c(T[p(x)]) \leq Z_c(p(x))$$

where $p(x)$ and $T[p(x)]$ are *real polynomials* and $Z_c(P(x))$ denotes the number of non-real zeros of $p(x)$, counting multiplicities.

Problem 12 ([BBS1)**.**] Let V be a cone in $\mathbb{R}^n$ and denote by $\mathcal{H}^n(V)$ the set of homogeneous polynomials in n variables that are (Gårding) hyperbolic with respect to any vector $e \in V$. Describe all linear operators T on $\mathbb{R}[z_1, \ldots, z_n]$ such that $T : \mathcal{H}^n(V) \to \mathcal{H}^n(V) \cup \{0\}$.

It is well known that the analog of the Lax Conjecture for hyperbolic polynomials in four or more variables fails to hold.

Problem 13 ("Stable" Lax Conjecture 1 ([HV)**.** , see also Conjecture 1 in [BBS1])] Let $P(x_0, x_1, \ldots, x_m)$ be a real homogeneous polynomial hyperbolic with respect to $c = (c_0, c_1, \ldots, c_m) \in \mathbb{R}^{m+1}$ and L be a real linear form in $x_0, x_1, \ldots, x_m$ with $L(c) \neq 0$. Then there exists an integer N such that

$$L(x_0, x_1, \ldots, x_m)^N P(x_0, x_1, \ldots, x_m) = \det(x_0 A_0 + x_1 A_1 + \ldots + x_m A_m)$$

for some real symmetric matrices $A_0, A_1, \ldots, A_m$ with $c_0 A_0 + c_1 A_1 + \ldots + c_m A_m > 0$.

As in [J-con] (see also § below) we let $A_1, \ldots, A_m$ be $n \times n$ matrices and define

$$\eta(A_1, \ldots, A_m) := \sum_{S_1, \ldots, S_m} \det(A_1[S_1]) \cdots \det(A_m[S_m]),$$

where the sum is over all m-tuples (partitions) of subsets $(S_1, \ldots, S_m)$ of $\{1, \ldots, n\}$ such that the S_i's are pairwise disjoint and $S_1 \cup \cdots \cup S_m = \{1, \ldots, n\}$. In [J-con] it was proved that the polynomial $\eta(L_1, \ldots, L_m)$ is real stable for all m-tuples of "positive" pencils L_j, where

$$L_j = L_j(z_1, \ldots, z_\ell) = \sum_{k=1}^{\ell} A_{jk} z_k + B_j, \tag{0.1}$$

and where for $1 \leq j \leq m$ the matrices $A_{jk}, 1 \leq k \leq \ell$, are positive semidefinite $n \times n$ matrices and B_j is Hermitian. In particular, the polynomial $\eta(z_1 A, \ldots, z_n A)$ is real stable, homogeneous and symmetric whenever A is positive semidefinite.

Problem 14 ("Stable" Lax Conjecture 2 (Problem 1 in [J-con)**.**)] Is it true that if f is a real stable polynomial of degree n in ℓ variables then there exist a positive integer m and matrix pencils L_j, $1 \leq j \leq m$, of the form (0.1) such that $f = \eta(L_1, \ldots, L_m)$?

The homogeneous version of Problem 14 is as follows.

Problem 15 ("Stable" Lax Conjecture 3 (Problem 2 in [J-con)**.**)] Let f be a real stable homogeneous polynomial of degree n in ℓ variables. Is it true that there exist a positive integer m and matrix pencils L_j, $1 \leq j \leq m$, of the form (0.1) with $B_j = 0$, $1 \leq j \leq m$, such that $f = \eta(L_1, \ldots, L_m)$?

Problem 16 (Determinantal representations for Laguerre-Pólya functions)**.** Let $\mathcal{LP}_2$ denote the Laguerre-Pólya class of entire functions in two variables (cf., e.g., [**?**, Chap. IX]; see also [BBS2]). Are there Lax type (infinite) determinantal representations for functions in $\mathcal{LP}_2$? For instance, is any function in $\mathcal{LP}_2$ given by the Weinstein-Aronszajn determinant (see, e.g., [**?**, Chap. IV.6] and [Bor-1]) of a pencil of infinite Hermitian/positive definite matrices?

Problem 17 (Analog of Hilbert's 17^{th} Problem)**.** Let $\mathcal{P}_n$ be the set of all real polynomials in n variables and $\mathcal{P}_n^+$ be the subset of $\mathcal{P}_n$ consisting of nonnegative polynomials (i.e., polynomials which are nonnegative whenever all variables are real). Denote by $\mathcal{S}_n$ the subset of $\mathcal{P}_n^+$ consisting of polynomials which are sums of squares of polynomials in $\mathcal{P}_n$ and let $\mathcal{M}_n$ be the set of all finite order linear partial differential operators with constant coefficients T on $\mathbb{R}_n[z_1,\ldots,z_n]$ such that $T(\mathcal{S}_n)\subseteq\mathcal{P}_n^+$. As is well known, $\mathcal{S}_1=\mathcal{P}_1^+$ but $\mathcal{S}_n\subsetneq\mathcal{P}_n^+$ as soon as $n\geq 2$. Is it true that $\mathcal{M}_n\cdot\mathcal{S}_n=\mathcal{P}_n^+$ for all n, that is, given $P\in\mathcal{P}_n^+$ there exists $S\in\mathcal{S}_n$ and $T\in\mathcal{M}_n$ such that $P=T(S)$?

Multiplier Sequences, CZDS and λ-Sequences.

Problem 18 ([CC18,CC20)**.**] A sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a *complex zero decreasing sequence (CZDS)*, if

$$Z_c\left(\sum_{k=0}^{n}\gamma_k a_k x^k\right)\leq Z_c\left(\sum_{k=0}^{n}a_k x^k\right),$$

for any real polynomial $\sum_{k=0}^{n}a_kx^k$. Characterize all complex zero decreasing sequences.

Problem 19 ([PS,BBS2,CPP)**.**] A sequence $T=\{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence* if, whenever the real polynomial $p(x)=\sum_{k=0}^{n}a_kx^k$ is hyperbolic, the polynomial $T[p(x)]:=\sum_{k=0}^{n}\gamma_ka_kx^k$ is also hyperbolic. (Thus, multiplier sequences can be viewed as hyperbolicity preserving linear operators.) Characterize the multiplier sequences which are CZDS (see Problem 18).

Problem 20 ([CC18)**.**] It is known that the sequence $\{e^{-k^p}\}_{k=0}^{\infty}$, where p is a positive integer, $p\geq 3$, is a multiplier sequence. Are these multiplier sequences CZDS (see Problem 18)?

Problem 21. With $\Phi(t)$ defined as in Problem 33 below, let b_k, $k\geq 0$, denote the k^{th} moment of Φ, defined by

$$b_k:=\int_0^{\infty}t^{2k}\,\Phi(t)\,dt,\quad\text{and}\quad\gamma_k:=\frac{k!}{(2k)!}b_k,\quad k=0,1,2,\ldots.$$

Then the **Riemann Hypothesis** holds if and only if $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence (cf. Problem 19) or that $\{\frac{\gamma_k}{k!}\}_{k=0}^{\infty}$ is a totally positive sequence. Another equivalent formulation is as follows. Set

$$F(x):=\sum_{k=0}^{\infty}\frac{b_k}{(2k)!}x^k=\int_0^{\infty}\Phi(t)\cosh(t\sqrt{x})\,dt,$$

Then the Riemann Hypothesis holds if and only if the sequence $\{F(k)\}_{k=0}^{\infty}$ is a CZDS. (For the definition of CZDS see Problem 18). In terms of the gamma and zeta functions, $F(k)$, $k=0,1,2\ldots$, can be expressed as

$$F(k)=\frac{\pi^{-1/4}}{64}\,(k-1)\,\pi^{-\sqrt{k}/4}\,\Gamma\left(\frac{1}{4}+\frac{\sqrt{k}}{4}\right)\zeta\left(\frac{1}{2}+\frac{\sqrt{k}}{2}\right).$$

Problem 22 ([CC18,CC20)**.**] A sequence of nonzero real numbers $\Lambda=\{\lambda_k\}_{k=0}^{\infty}$ is called a *λ-sequence* (or *positive definite sequence*) if

$$\Lambda[p(x)]:=\Lambda\left[\sum_{k=0}^{n}a_kx^k\right]:=\sum_{k=0}^{n}\lambda_ka_kx^k>0\quad\text{ for all }x\in\mathbb{R},$$

whenever $p(x)=\sum_{k=0}^{n}a_kx^k>0$ for all $x\in\mathbb{R}$. Characterize those multiplier sequences $\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k>0$, for which the sequences of reciprocals, $\{1/\gamma_k\}_{k=0}^{\infty}$, are λ-sequences.

Problem 23 ([BCC,BCCG).] Let $\varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$, $\gamma_k \geq 0$, denote a transcendental real entire function of exponential type with only real negative zeros. If $\overline{\lim}_{k\to\infty} \gamma_k^{1/k} = 0$, then is the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a CZDS (cf. Problem 18)?

Problem 24 ([CC17b).] Characterize the meromorphic functions $F(x) = \varphi(x)/\psi(x)$, where φ and ψ are entire functions such that the polynomial $\sum_{k=0}^{n} F(k) a_k x^k$ has only real zeros whenever the polynomial $\sum_{k=0}^{n} a_k x^k$ has only real zeros.

Problem 25 ([CC17b).] Characterize the meromorphic functions $F(x)$ with the property that $\sum_{k=0}^{\infty} F(k) a_k x^k/k!$ is a transcendental entire function with only real zeros (or the zeros all lie in the half-plane $\mathfrak{Re} z < 0$) whenever the entire function $\sum_{k=0}^{\infty} a_k x^k/k!$ has only real zeros.

Problem 26 ([CC17b,Ka).] A multiplier sequence (see Problem 19 for the definition) of the form $\{F(k)\}_{k=0}^{\infty}$, where $F(x)$ is a meromorphic function, is called *a meromorphic Laguerre multiplier sequence*. If $\{F(k)\}_{k=0}^{\infty}$ is a meromorphic Laguerre multiplier sequence, is the sequence $\{F(k)\}_{k=0}^{\infty}$ a CZDS (cf. Problem 18)?

Special Functions and Polynomials with Real Zeros.

Problem 27 ([CC17b). , p. 90 in [Ka]] The *Wright function* (or *Fox-Wright function)* is defined as

$$ {}_p\Psi_q(x) := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j k + b_j)}{\prod_{j=1}^{q} \Gamma(c_j k + d_j)} \frac{x^k}{k!}, \tag{0.2} $$

where $\Gamma(x)$ denotes the gamma function and p and q are nonnegative integers. If we set $b_j = 1$ $(j = 1, 2, 3 \ldots, p)$ and $d_j = 1$ $(j = 1, 2, 3 \ldots, q)$, then (0.2) reduces to the familiar *generalized hypergeometric function*

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}, $$

where the *Pochhammer* or ascending factorial symbol for $a \in \mathbb{C} \setminus \{0\}$ is defined as $(a)_0 = 1$, $(a)_k := a(a+1)(a+2)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$, $k = 1, 2, 3 \ldots$. (Today, notwithstanding the prolific research activity in this area, there is little known about the distribution of zeros of the Wright function and the generalized hypergeometric function, except in some very special cases.) Obtain information about the distribution of zeros of those Wright functions and generalized hypergeometric functions which are entire functions.

Problem 28 ([CC17b). , p. 90 in [Ka]] Consider

$$ {}_2\Psi_1(x) = \sum_{k=0}^{\infty} \frac{\Gamma(ak+1)\Gamma(bk+1)}{\Gamma(ck+d+1)} \frac{x^k}{k!}, \quad a, b, c, d \geq 0 \quad \text{and} \quad c \geq a + b. $$

Under what additional restrictions on the parameters a, b, c, d is it true that the function ${}_2\Psi_1(x)$ has only real negative zeros?

Problem 29 ([CC17b). , p. 90 in [Ka]] In Problem 28, set $a = m$, $b = 0$, $c = m + 1$ and $d = -1$, where m is a nonnegative integer. Then we obtain the sequence of (entire) Wright functions:

$$f_m(x) := \sum_{k=0}^{\infty} \frac{\Gamma(mk+1)}{\Gamma((m+1)k+1)} \frac{x^k}{k!}, \quad m = 0, 1, 2 \ldots.$$

It is known that the entire functions $f_0(x) = J_0(2\sqrt{x})$ (Bessel function) and $f_1(x) = \cosh(\sqrt{x})$ have only real zeros. Note that for arbitrary positive integers $m \geq 2$, $f_m(x)$ is an entire function of order 1/2. Does $f_m(x)$, $m \geq 2$, have only real zeros?

Problem 30 ([CCW).] The *Jacobi polynomial*, $P_n^{(\alpha,\beta)}(x)$, of degree n, is defined by

$$P_n^{(\alpha,\beta)}(x) := \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x-1)^{n-k}(x+1)^k, \quad \alpha,\ \beta > -1.$$

If $n \geq 2$, $-1 < \alpha \leq 1$ and $\beta > -1$, then it is known ([**?**, Theorem 2]) that the polynomials

$$\varphi_n(x) := \sum_{k=0}^{[n/2]} \left(\frac{d^{2k}}{dx^{2k}} P_n^{(\alpha,\beta)}(x) \right)_{x=1} x^k \qquad \text{and}$$

$$\psi_n(x) := \sum_{k=0}^{[n/2]} \left(\frac{d^{2k+1}}{dx^{2k+1}} P_n^{(\alpha,\beta)}(x) \right)_{x=1} x^k$$

have only real, simple negative zeros. We conjecture that for $n \geq 4$ the zeros of $\varphi_n(x)$ and $\varphi_{n-1}(x)$ interlace and that the zeros of $\psi_n(x)$ and $\psi_{n-2}(x)$ interlace (cf. [**?**, Conjectures 2 and 3] and S. Fisk [Fis]).

Problem 31 (Problem 18 in [CCW).] Characterize the class of all real polynomials, $p_n(x)$, of degree n, all of whose zeros lie in the interval $(-1, 1)$, such that that the associated polynomials $\varphi_n(x) := \sum_{k=0}^{[\frac{n}{2}]} p_n^{(2k)}(1)x^k$ possess only real negative zeros.

Dynamics of Zeros under Fourier Transforms and Related Operators.

In [CY23d] it was pointed out that today, there are no known necessary *and* sufficient conditions that a "nice" kernel, $K(t)$, must satisfy in order that its Fourier transform have only real zeros. It is this fundamental issue that motivates some of the questions and results dealing with the distribution of zeros of real entire functions represented by Fourier transforms. By way of background information, we mention Pólya's approach to this problem via *universal factors* ([**?**, pp. 265–277], see also [**?**, pp. 166–197] and [dB]). We recall that an entire function f is a *universal factor* if the entire function

$$\int_{-\infty}^{\infty} e^{izt} K(t) f(it)\, dt$$

has only real zeros whenever the zeros of the Fourier transform of $K(t)$ are all real, where $K : \mathbb{R} \to \mathbb{R}$ is integrable over $\mathbb{R}$, $K(t) = K(-t)$ for all $t \in \mathbb{R}$ and $K(t) = O(\exp(-|t|^{2+\varepsilon})$ for some $\varepsilon > 0$, as $t \to \pm\infty$. Pólya has shown [**?**, pp. 265–277] that $f(it)$ is a universal factor if and only if f is a real entire function (in the Laguerre–Pólya class) of order at most 2 having only real zeros.

Problem 32 (Open Problem 4.8 in [CC18b).] In [dB] de Bruijn proved that if f is a real entire function of order less than two and if all the zeros of f lie in the strip $S(\Delta) = \{\, z \in \mathbb{C} \mid |\mathfrak{Im} z| \leq \Delta \,\}$ $\quad (\Delta \geq 0)$, then the zeros of $\cos(\lambda D)f(x)$ $\quad (\lambda \geq 0$, D=d/dx) satisfy

$|\mathfrak{Im} z| \leq \sqrt{\Delta^2 - \lambda^2}$, if $\Delta > \lambda$ and $\mathfrak{Im} z = 0$, if $0 \leq \Delta \leq \lambda$. (This result may be viewed as an analog of Jensen's theorem on the location of the non-real zeros of the derivative of a polynomial.) Is there also an analog of Jensen's theorem for $\varphi(\lambda D)f(x)$ when φ is a more general transcendental entire function? (See also [ABG,Bor,CCS21,FR].)

Problem 33 (D. A. Cardon [Car)**.** , H. Ki, Y.-O. Kim [KK28b]; see also [C23a,CV23b,CV23c,CY23d]] Consider the finite Fourier transform

$$H_R(x) := \int_0^R \Phi(t)\cos(xt)\,dt,$$

where the *Jacobi theta function*, $\Phi(t)$, is defined by

$$\Phi(t) := \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}\right) \exp\left(-\pi n^2 e^{4t}\right).$$

If R is sufficiently large, are the zeros of the $H_R(x)$ located in the horizontal strip $|\mathfrak{Im} z| < 1$? For what values of $R > 0.11$, if any, does $H_R(x)$ possess some non-real zeros? (For $0 < R < 0.11$ it is known that the entire function $H_R(x)$ has only real zeros [**?**, Theorem 3.6]). This latter problem can also be expressed in terms of multiplier sequences (cf. Problem 19).

Problem 34. We recall [Le] that if $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function, then its *order* $\rho = \rho(\varphi) = \overline{\lim}_{n\to\infty} \frac{n \log n}{-\log|a_n|}$. Set $\gamma_k = k^{\sqrt{k}}$ for $k = 1, 2, 3 \ldots$. Then for each $m \geq 1$, the real entire function $f_m(x) := \sum_{k=0}^{\infty} \frac{\gamma_{k+m}}{k!} x^k$ is of order 1. Note that the m^{th} derivative of $f_1(x)$ is $f_{m+1}(x)$. For all m sufficiently large, is the sequence $\{\gamma_{k+m}\}_{k=1}^{\infty}$ a multiplier sequence (see Problem 19 for the definition of a multiplier sequence)? This problem arises in connection with the investigation of the distribution of zeros of entire functions of order 1. For example, the function $H(x) := \int_0^{\infty} \Phi(t)\cos(xt)\,dt$ (cf. Problem 33) is an entire function of order 1 and of maximal type. (See also D. W. Farmer, R. C. Rhoades [FR].)

Symmetric, Real Stable and Homogeneous Polynomials and the POT Conjecture.

Let $A_1, \ldots, A_m$ be $n \times n$ matrices, and define

$$\eta(A_1, \ldots, A_m) := \sum_{S_1, \ldots, S_m} \det(A_1[S_1]) \cdots \det(A_m[S_m]),$$

where the sum is over all m-tuples (partitions) of subsets $(S_1, \ldots, S_m)$ of $\{1, \ldots, n\}$ such that the S_i's are pairwise disjoint and $S_1 \cup \cdots \cup S_m = \{1, \ldots, n\}$. In [J-con] it was proved that the polynomial $\eta(L_1, \ldots, L_m)$ is real stable for all m-tuples of "positive" pencils L_j, where

$$L_j = L_j(z_1, \ldots, z_\ell) = \sum_{k=1}^{\ell} A_{jk} z_k + B_j,$$

and where for $1 \leq j \leq m$ the matrices $A_{jk}, 1 \leq k \leq \ell$, are positive semidefinite $n \times n$ matrices and B_j is Hermitian. In particular, the polynomial $\eta(z_1 A, \ldots, z_n A)$ is real stable, homogeneous and symmetric whenever A is positive semidefinite.

A most fascinating fact [BB] (which can be proved using results from [goulden-jackson]) is that

$$\eta(z_1 A, \ldots, z_n A) = \sum_{\lambda \vdash n} \mathrm{Imm}_{\lambda'}(A) s_\lambda(z_1, \ldots, z_n),$$

where s_λ is the *Schur-function* indexed by the partition λ, $\mathrm{Imm}_\lambda(A)$ is the *immanant* indexed by λ, and λ' is the conjugate partition. This raises many questions.

Problem 35 ([BB)**.**] Characterize all symmetric, real stable and homogeneous polynomial of degree d in n variables. Is it in fact so that if p is such a polynomial, then there is a $d \times d$ positive semidefinite $d \times d$ matrix A such that

$$p(z_1, \dots, z_n) = \eta(z_1 A, \dots, z_n A)?$$

This is not obvious even for $n = 2$.

Problem 36 ([BB)**.**] Let p be a symmetric, real stable and homogeneous polynomial of degree d in n variables with $d \leq n$ and suppose that p has at least one positive (and then all non-negative) coefficients in the monomial basis. Is p Schur-positive? That is, when $p(z_1, \dots, z_n)$ is expanded in the Schur-basis

$$p(z_1, \dots, z_n) = \sum_{\lambda \vdash d} a_\lambda s_\lambda(z_1, \dots, z_n),$$

is $a_\lambda \geq 0$ for all $\lambda \vdash d$?

If A is a $d \times d$ matrix then $\det(A) = \mathrm{Imm}_{(1^d)}(A)$ and $\mathrm{per}(A) = \mathrm{Imm}_{(d)}(A)$, where $(1^d) = (1, \dots, 1)$ and $(d) = (1^d)' = (d, 0, \dots, 0)$. Schur's inequality [J,L] asserts that if A is a $d \times d$ positive semi-definite matrix then $\mathrm{Imm}_\lambda(A) \geq f^\lambda \det(A)$ for all $\lambda \vdash d$, where f^λ is the number of standard Young tableaux of shape λ. A natural real stable extension of this result would be as follows.

Problem 37 (Stable Schur's Inequality, [BB)**.**] Let p be as in Problem 36. Is $a_\lambda \geq f^\lambda a_{(d)}$ for all $\lambda \vdash d$?

Lieb's Permanent-On-Top (POT) Conjecture for the symmetric group on d elements [J,L] asserts that if A is a $d \times d$ positive semi-definite matrix then $\mathrm{Imm}_\lambda(A) \leq f^\lambda \mathrm{per}(A)$ for all $\lambda \vdash d$. An affirmative answer to the following problem would in particular imply the validity of the POT Conjecture.

Problem 38 (Stable Permanent-On-Top Inequality, [BB)**.**] Let p be as in Problem 36. Is $a_\lambda \leq f^\lambda a_{(1^d)}$ for all $\lambda \vdash d$?

Stable and Hyperbolic BMV Conjectures.

The well-known Bessis-Moussa-Villani (BMV) Conjecture [BMV] would considerably simplify the calculation of partition functions of quantum mechanical systems. As shown in [L-S], the BMV Conjecture may be reformulated as follows.

Problem 39 (BMV Conjecture)**.** Let $n \in \mathbb{N}$ and A, B be arbitrary positive definite $n \times n$ matrices. Then for any $k \in \mathbb{N}$ the polynomial $z \mapsto \mathrm{Tr}\big((A + zB)^k\big)$ has only non-negative coefficients.

By using the Lax Conjecture/Theorem and its analog for real stable polynomials in two variables ([**?**, Theorem 23 and Corollary 4]) one can show that Problems 40–41 below are equivalent to the BMV Conjecture.

Problem 40 ([BB,BBS).] Suppose that $f \in \mathbb{R}[z_1, z_2]$ is real stable with $f(0,0) = 1$ and that all Taylor coefficients of f are non-negative. Is it true that the function

$$(z_1, z_2) \mapsto -\log(f(-z_1, -z_2))$$

has all non-negative Taylor coefficients?

Problem 41 ([BB,BBS).] Let p be an e-hyperbolic polynomial for which $p(e) = 1$, where $e \in \mathbb{R}^n$, and suppose that v_1, v_2 are in the positivity cone $C_p(e)^+$ of p w.r.t. e. Are all Taylor coefficients of

$$(z_1, z_2) \mapsto -\log(p(e - z_1 v_1 - z_2 v_2))$$

non-negative?

It is known that the analog of the BMV Conjecture for three or more positive definite matrices is not true. Therefore, both the analog of Problem 40 for real stable polynomials in three or more variables and the analog of Problem 41 for three or more vectors in the positivity cone $C_p(e)^+$ are false. Note that as we already pointed out in §, the analog of the Lax Conjecture for hyperbolic polynomials in four or more variables also fails to hold.

Rayleigh-type Correlations, Newton's Inequalities and Negative Dependence Theory.
Recall that a multivariate polynomial is *multi-affine* if it has degree at most one in each variable. A multi-affine polynomial $f \in \mathbb{R}[z_1, \dots, z_n]$ with non-negative coefficients is called a *Rayleigh polynomial* [W] if

$$\frac{\partial f}{\partial z_i}(x)\frac{\partial f}{\partial z_j}(x) \geq \frac{\partial^2 f}{\partial z_i \partial z_j}(x) f(x),$$

for all $x \in \mathbb{R}_+^n$ and $1 \leq i < j \leq n$. Recall that a sequence $(a_k)_{k=0}^n$ is *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k \leq n-1$. The following conjecture in negative dependence theory was initially formulated – albeit in a different form – by Pemantle [P]. Wagner stated this same conjecture as follows in [W], where it was baptized the "Big Conjecture".

Conjecture 1 ([P,W).] If $f \in \mathbb{R}[z_1, \dots, z_n]$ is Rayleigh and

$$\Delta(f)(t) := \sum_{k=0}^{n} \binom{n}{k} c_k t^k$$

is the diagonal specialization of f then the sequence $(c_k)_{k=0}^n$ is log-concave with no internal zeros.

Conjecture 1 was recently disproved in [BBL], where counterexamples were constructed for all $n \geq 20$.

Problem 42 ([BBL).] Describe in terms of Rayleigh-type correlations a natural class $\mathcal{RN}_n$ of multi-affine polynomials in n variables with non-negative coefficients such that if $f \in \mathcal{RN}_n$ and

$$\Delta(f)(t) = \sum_{k=0}^{n} \binom{n}{k} c_k t^k$$

is the diagonal specialization of f then the sequence $(c_k)_{k=0}^n$ is log-concave with no internal zeros.

Remark 2. As noted in [BBL], all real stable multi-affine polynomials in n variables with non-negative coefficients satisfy both the hypothesis and the conclusion of Conjecture 1.

Given an $n \times n$ matrix A and a subset S of $\{1, \dots, n\}$ let $A\langle S\rangle = \det(A[S'])$ be the principal minor of A with rows and columns indexed by $S' = \{1, \dots, n\} \setminus S$. A square matrix is a *$P$-matrix* if all principal minors of A are positive. One says that a P-matrix A is a *GKK-matrix* (after Gantmacher, Kotelijansky and Krein) if it satisfies the *Hadamard-Fischer-Kotelijansky inequalities*, that is,

$$A\langle S\rangle A\langle T\rangle \geq A\langle S \cup T\rangle A\langle S \cap T\rangle, \quad S, T \subseteq \{1, \dots, n\}.$$

Let $z_1, \dots, z_n$ be commuting variables and set $Z = \operatorname{diag}(z_1, \dots, z_n)$. To each $n \times n$ matrix A we associate a multi-affine polynomial

$$f_A(z) = \det(A + Z) = \sum_{S \subseteq \{1,\dots,n\}} A\langle S\rangle z^S, \quad z = (z_1, \dots, z_n),\ z^S := \prod_{i \in S} z_i.$$

We say that a P-matrix A is a *Rayleigh matrix* if $f_A(z)$ is a Rayleigh polynomial. It follows from [**?**, Theorem 4.4] that Rayleigh matrices are GKK. In fact, the following holds.

Theorem 2 ([BBL)**.** *] Let A be an $n \times n$ matrix over $\mathbb{R}$. The following are equivalent:*

(1) *A is a Rayleigh matrix,*
(2) *$A + X$ is a GKK-matrix for all $X = \operatorname{diag}(x_1, \dots, x_n)$, where $x_i \geq 0$ for all $1 \leq i \leq n$.*

An $n \times n$ matrix is an *M-matrix* if all its principal minors are non-negative and all off-diagonal entries are non-positive. Hence, non-singular M-matrices and positive semidefinite matrices are Rayleigh. In [HS] it was conjectured that M-matrices satisfy the conclusion of Conjecture 1. This was subsequently proved in [HM].

Problem 43 ([BBL)**.**] Does the polynomial $f_A(z)$ satisfies the conclusion of Conjecture 1 whenever A is Rayleigh?

Problem 44 (Rayleigh matrices, [BBL)**.**] Describe all $n \times n$ Rayleigh matrices.

Problem 45 (Real Stable Matrices, [BBL)**.**] Characterize the class of all real stable $n \times n$ matrices, that is, $n \times n$ matrices A such that $f_A(z)$ is real stable.

Miscellaneous Problems.

Problem 46 ([CC20b)**.** , see also [Bor,CC17,CC19,CD22,Gar,Pit,PS,St]] For any real entire function $\varphi(x)$, set

$$\mathcal{T}_k^{(1)}(\varphi(x)) := (\varphi^{(k)}(x))^2 - \varphi^{(k-1)}(x)\varphi^{(k+1)}(x) \quad \text{if} \quad k \geq 1,$$

and for $n \geq 1$, set

$$\mathcal{T}_k^{(n)}(\varphi(x)) := (\mathcal{T}_k^{(n-1)}(\varphi(x)))^2 - \mathcal{T}_{k-1}^{(n-1)}(\varphi(x))\, \mathcal{T}_{k+1}^{(n-1)}(\varphi(x)) \quad \text{if} \quad k \geq n \geq 1.$$

(Note that with the notation above, we have $\mathcal{T}_{k+j}^{(n)}(\varphi) = \mathcal{T}_k^{(n)}(\varphi^{(j)})$ for $k \geq n$ and $j = 0, 1, 2 \dots.$) If $\varphi(x)$ is a real entire function of order at most one with only real negative zeros, are the *iterated Laguerre inequalities* valid for all $x \geq 0$? That is, is it true that

$$\mathcal{T}_k^{(n)}(\varphi(x)) \geq 0 \qquad \text{for all} \quad x \geq 0 \quad \text{and} \quad k \geq n?$$

Problem H (The Hawai'i Conjecture, see p. 429 in [CCS21)**.** , T. Sheil-Small [ShS]] Let $p(x)$ be a real polynomial of degree $n \geq 2$, and suppose that $p(x)$ has exactly $2d$ nonreal zeros, $0 \leq 2d \leq n$. Let

$$Q(x) := D\left(\frac{p'}{p}(x)\right), \quad \text{where} \quad D := \frac{d}{dx}.$$

Then the Hawai'i conjecture asserts that

$$Z_R\left(Q(x)\right) \leq 2d,$$

where $Z_R(Q)$ denotes the number of real zeros of Q, counting multiplicities.

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