Polyhedral geometry and partition theory

organized by
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Workshop Summary

We held a workshop at AIM in November of 2016, which brought together a diverse group of researchers to study problems at the interface of polyhedral geometry and partition theory. These two areas, with a few exceptions related to work of Richard Stanley, until recently found quite separate communities. In view of recent interdisciplinary progress, the overall goal of the proposed workshop was to bring researchers from both of these worlds together to exchange ideas, address problems, and learn techniques across the areas. This workshop exceeded our expectations.

Background and Goals

This idea for a new AIM workshop grew out of a successful AIM SQuaRE on *Partition Theory and Polyhedral Geometry*, which held a series of one-week meetings at AIM in November of 2010, 2011, and 2012. A major theme of this previous work was to uncover geometric interpretations of important theoretic results. Our goal in the November 2016 workshop was to further investigate these connections, with an emphasis on the following three topics:

- the geometric and algebraic structure of lecture hall partitions; their relationship to permutation and Coxeter groups, (rational) Catalan combinatorics, and hyperplane arrangements
- the geometry, combinatorics, and computation of vector partition functions; the interpretation and application of recent structural results; the discovery of new formulas
- unimodality/real rootedness questions in Ehrhart theory, partition theory, and Coxeter groups

These three focused research topics are important because they each have the potential to be a unifying framework for a wide range of ideas in combinatorics, and they are closely related to each other.

We began the workshop by telling participants that our goal was to host a workshop that:

(1) provides everyone with a good idea of the state of the art of the field
(2) unearths and clarifies what we see collectively as the most interesting future directions and problems
(3) makes progress towards some of these directions
(4) benefits from the diversity of experiences and expertise of all the participants
(5) embraces the AIM philosophy of collaborative, not competitive, work in mathematics
(6) makes sure everyone benefits and is able to contribute, and

(7) gives every participant some interesting research projects to bring back home.

Our goal was to be explicit about the positive, dynamic, collaborative atmosphere that we were seeking to create, and several participants commented on the success of this effort.

Talks were presented in the morning sessions by Carla Savage, Federico Ardila, Benjamin Braun, Peter Paule, Zafeirakis Zafeirakopoulos, Alejandro Morales, Karola Meszaros, Petter Branden, Jesus de Loera, and Matthias Koeppe. Additionally, there was a morning session devoted to presentations about mathematical software, presented by Koeppe, Zafeirakopoulos, and Paule. In the following sections, we give more details regarding the problem groups from the workshop.

Roots of Eulerian Polynomials

We started with a conjecture Kyle Petersen made about the asymptotics of the largest roots of Eulerian polynomials. Namely, the largest root grows like 2^n, the second largest like $(3/2)^n$, the third largest like $(4/3)^n$, and so on. By Wednesday afternoon, Silviu Radu had the idea for proving the largest root case, which the whole group helped verify with a careful induction argument. By Friday, the group had generalized Silviu's idea to give a proof for the kth largest root. Moreover, the ingredients of the proof were very general. It was quickly realized that the key ingredient (possessed by the Eulerian polynomials) was what we call "super log-concavity", namely that for all $j \leq k$, the ratio $a_j^2/(a_{j-1}a_{j+1}$ tends to infinity as $n \to \infty$, where $a_j = a_{j,n}$ is the coefficient of x^{n-j} in the nth polynomial. This super log-concavity is possessed by the Eulerian polynomials of types A and B, but also for any h^*-polynomial whose Ehrhart polynomial is exponential in n.

Volumes and Lattice Points of Flow Polytopes

We studied two intriguing facts: the volume of the flow polytope $F(1, 0, \dots, 0, -1)$ and the number of lattice points of the flow polytope $F(1, 2, 3, \dots, n-1, -n(n-1)/2)$ both equal a product of initial terms of the Catalan sequence. All the known proofs of these results are analytic, and we seeked a combinatorial/geometric understanding of these facts.

On the volume side, we observed that the flow polytope is compressed, so it has nice triangulations which could make the volume computation into a combinatorial problem. We tried to understand the combinatorial results of participants Karola Meszaros and Alejandro Morales geometrically, in terms of these triangulations, with partial but promising results. Ultimately, the hope is that the geometric-combinatorial properties of the pulling triangulations of this flow polytope will reveal their "Catalan inductive structure."

On the lattice point side, the group studied a more general family of "Caracol flow polytopes" which seem to have nice enumerative properties, using some techniques of participant Pamela Harris in her work on Kostant partition functions. We continue to discuss these two projects.

Linear Factors of Kostant Partition Functions

Kostant partition functions $K(a_1, \dots, a_n)$ are piecewise polynomial in the a_is, and these polynomials often have many linear factors. Federico Ardila brought a possible explanation to the workshop and, by talking to the experts on computing these polynomials gathered at AIM, he found that this technique explains all the linear factors that have been observed experimentally. In the most extreme cases, a polynomial of degree $\sim n^2/2$ is shown to have

$\sim n^2/4$ linear factors. This could be a useful tool to explain the formulas discussed in the previous group.

Roots of Durfee Polynomials

Let $d(\lambda)$ be the size of the Durfee square of a partition λ. Define $D_n(x) := \sum_{\lambda\vdash n} x^{d(\lambda)}$ to be the Durfee polynomial of n. This group is investigating the real-rootedness of $D_n(x)$; it is known to be real-rooted for values of n up to 50 (checked by a computer). During the workshop, the group attempted to solve this problem by considering a recursive formula of the coefficients of $D_n(x)$. The group also considered polynomials $h_n(x)$ such that $D_n(x)$ is the h_n-Eulerian polynomial in order to use a result of Brenti to conclude that $D_n(x)$ is real-rooted. So far, the group does not have any concrete results. We will continue to meet over Skype in order to continue work on this project.

Hilbert Bases for s-Lecture Hall Cones

The problem was to determine the Hilbert basis of the cone $\{x \in \mathbf{R}^n : 0 \leq \frac{x_1}{s_1} \leq \frac{x_2}{s_2} \leq \cdots \leq \frac{x_n}{s_n}\}$ for various sequences of positive integers $(s_1, s_2, \ldots)$. We did computer experiments in the case $n = 2$ and found that there is substantial variation in the behavior of the Hilbert bases for these cones, even in two dimensions. The decision was made to investigate the behavior of the Hilbert bases in the case where the cones are Gorenstein. McCabe Olsen is continuing this project following the workshop.

Eulerian transformation of polynomials

Our group was working on the following problem proposed by Francesco Brenti:

- Let $\sum_{i=0}^{n} b_i x^i$ be a real-rooted polynomial with $b_0, \ldots, b_n \geq 0$. Then also $\sum_{i=0}^{n} b_i A_i(x)$ is real-rooted.

During the workshop we proved that the transformation $x^i \mapsto S_i(x) := \sum_{l=0}^{i} S(i,l)x^l$ preserves real-rootedness, where $S(i,l)$ denote the Stirling numbers of the second kind. Since the polynomials $S_i(x)$ are closely related to the Eulerian polynomials $A_i(x)$ we gained some hope. However, shortly after the workshop concluded, our group member Petter Brändén found a counterexample to the general conjecture, namely $(1 + \frac{x}{5})^5$. During the workshop we developed an intruiging subconjecture which is still open and which states that

$$\left\{ \sum_{j=0}^{n-i} \binom{n-i}{j} A_{i+j}(x) \right\}_{0 \leq i \leq n}$$

forms an interlacing family of polynomials, and by now we have computational evidence for $n \leq 26$. If true, this would prove that the image under the transformation $x^i \mapsto A_i(x)$ of all polynomials of the form $\sum_{i=0}^{n} b_i x^i (1+x)^{n-i}$ and $b_0, \ldots, b_n \geq 0$ is real-rooted.

Coxeter interpretations of s-Lecture Hall Partitions

Our group began by reviewing the abacus diagrams that are in bijection with the type C minimal length coset representatives, and hence with the classical Lecture Hall Partitions. F. Brenti derived a direct proof that the abacus construction encodes the Lecture Hall inequalities and we spent some time brainstorming ways to generalize this result, particularly for inequalities arising from ℓ-sequences. We then investigated a 2-variable generating function that would decompose the Coxeter length statistic into even and odd parts. At the end of the session, we investigated a potential connection to quasisymmetric functions via Savage's work on the Lecture Hall Partitions using Gessel's barred permutations.

Efficient Computation of Vector Partition Functions

A group investigated computational techniques for computing vector partition functions effectively.

Combinatorial Proof of the Morris Identity

A group investigated various approaches to a possible combinatorial proof of the Morris identity, which has played a key role in the proofs of the volume formula for the Chen-Robbins-Yuen polytopes related to Catalan numbers.