QUESTIONS PROPOSED AT THE A.I.M. WORKSHOP

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Abstract. An AIM workshop on "Projective modules and $\mathbb{A}^1$-homotopy theory" was organized at the American Institute of Mathematics from the 5^{th} to the 9^{th} of May. Participants were asked to propose questions that they were interested in and felt are important to furthering the subject. Below is a list of these questions and some related discussions.

Question 1

Question (Hailong Dao). *Let X be the punctured spectrum on a local ring $(R, \mathfrak{m})$. What is the obstruction theory for splitting of vector bundles over such a scheme?*

More specifically, let $X = \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ for $(R, \mathfrak{m})$ local of dimension d. What is the obstruction to splitting of rank $d-1$ vector bundles on X?

Related Question (Satya Mandal). *Let X be a scheme and $\mathcal{E}$ be a vector bundle. What is the obstruction to obtaining a surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_X$?*

Related Question (Aravind Asok). *In the affine case, one has a fair idea of the obstruction theory for projective modules. What is the obstruction theory for general classes of modules?*

Related Question (Hailong Dao). *Is there an Euler class group for reflexive modules over a regular local ring $(R, \mathfrak{m})$?*

Related Question. *What is the role of $\mathbb{A}^1$-homotopy theory in the non-affine case?*

Remark (Paul Balmer). Maybe one can use the Jounalou trick of affine replacement of non-affine varieties.

Remark (Aravind Asok). Let $\nu_r(X)$ denote the set of rank r vector bundles over a scheme X. There exists a non-affine variety Y and a map $f : \mathbb{A}^n \to Y$ which is an $\mathbb{A}^1$-weak equivalence but $\nu_r(Y)$ is large. Thus, for non-affine schemes, ν_r is not an $\mathbb{A}^1$-homotopy invariant.

Related Question (Aravind Asok). *Let V be a finite dimensional k-vector space. Let H be the codimension 1 hypersurface in $\mathbb{P}(V) \times \mathbb{P}(V^*)$ given by the incidence relation. Then*

$$\mathbb{P}(V) \times \mathbb{P}(V^*) \smallsetminus H \overset{p_1}{\twoheadrightarrow} \mathbb{P}(V)$$

is an $\mathbb{A}^1$-weak equivalence. Is p_1^* surjective on isomorphism classes of vector bundles?

Related Question (Kirsten Wickelgren). *What is the role of Nišnevich / étale topology with realization to equivariant topological vector bundles in manifolds?*

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Question 2

Question (by popular demand)**.** *What is the connection between Euler class groups and Chow-Witt groups?*

Remark (Jean Fasel). Given a variety X over $\mathbb{C}$, there is a cycle class map

$$CH^r(X) \xrightarrow{c^r} \mathrm{H}^{2r}(X(\mathbb{C}), \mathbb{Z}).$$

and given a variety X over $\mathbb{R}$, there is a cycle class map

$$CH^r(X) \xrightarrow{c^r} \mathrm{H}^r(X(\mathbb{R}), \frac{\mathbb{Z}}{2\mathbb{Z}}).$$

Both these maps are compatible with products.

Question (Jean Fasel)**.** *Let X be a variety over $\mathbb{R}$. Is there a map $\tilde{c}^r$ making the following diagram commute :*

$$\begin{array}{ccc} \widetilde{CH}^r(X, \omega_{X/\mathbb{R}}) & \overset{\tilde{c}^r}{\dashrightarrow} & \mathrm{H}^r(X(\mathbb{R}), \mathbb{Z}) \\ \downarrow & & \downarrow \\ CH^r(X) & \xrightarrow{c^r} & \mathrm{H}^r(X(\mathbb{R}), \frac{\mathbb{Z}}{2\mathbb{Z}}) \end{array}$$

which is also compatible with products?

Question 3

Question (Christian Haesemeyer)**.** *What are "motivic local systems"? Can one define motivic Atiyah classes?*

Question 4

Question (Kronecker-proposed by Satya Mandal)**.** *Are curves in $\mathbb{A}^n_k$ set theoretic complete intersection ideals?*

Question 5

Question (Satya Mandal)**.** *Let $I \subseteq k[X_1, X_2, \ldots, X_n] = R$ be an ideal. Then is*

$$\mu(I) = \mu(\frac{I}{I^2})?$$

Remark (Satya Mandal). This is known when $\mu(\frac{I}{I^2}) \geq \dim(\frac{R}{I}) + 2$.

Related Question (S.M.Bhatwadekar)**.** *More specifically, let α be a surjective k-algebra homomorphism $\mathbb{C}[X_1, X_2, \ldots, X_5] \overset{\alpha}{\twoheadrightarrow} \mathbb{C}[Y_1, Y_2]$ and let $I = ker(\alpha)$. We know $\mu(\frac{I}{I^2}) = 3$. Is $\mu(I) = 3$?*

Remark (Aravind Asok). One can use $\mathbb{A}^1$-homotopy to compute $\mu(P)$ for a projective module P. This is by considering if the map $X \to BGL_r$ lifts to some suitable Grassmannian.

Related Question (Jean Fasel)**.** *Let $X = \mathrm{Spec}(R)$ be smooth and $Y \subseteq X$ be a smooth embedding. Is there a cohomological obstruction (computable á la Chow groups) to Y being a complete intersection?*

Remark (Jean Fasel). Let

$$R_{univ} = \frac{k[X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, Z]}{(\sum_{i=1}^{n} X_i Y_i - Z(1+Z))}$$

$$X_{univ} = \mathrm{Spec}(R_{univ}) \qquad Y_{univ} = \mathrm{Spec}(\frac{R_{univ}}{(X_1, X_2, \ldots, X_n, Z)}).$$

Then the above structure has the universal property that given a k-variety $X = \mathrm{Spec}(R)$ and a subvariety $Y = \mathrm{Spec}(\frac{R}{I})$ with $\mu(\frac{I}{I^2})$, there is a natural map $X \xrightarrow{f} X_{univ}$ which restricts to a map $Y \xrightarrow{f} Y_{univ}$. The universal X_{univ} is a sphere in $\mathbb{A}^1$-topology.

$$\widetilde{CH^*}(X_{univ},) \xrightarrow{\sim} GW(k) \oplus GW(k)\alpha$$

where the element α is a form on the subvariety Y_{univ}.

Question (Jean Fasel). *With the above notations, given a k-variety X and a local complete intersection subvariety Y, let $f^*(\alpha)$ be the image of α under the induced map*

$$\widetilde{CH^*}(X_{univ},) \xrightarrow{f^*} \widetilde{CH^*}(X, \omega_{X/k}).$$

Is $f^*(\alpha) = 0$ sufficient for Y to be a complete intersection?

Question 6

Question (Anastasia Stavrova). *Can $\mathbb{A}^1$-homotopy be extended over general smooth schemes, e.g. over a dvr? Else explain the issues.*

Remark (Aravind Asok). Many basic things fail, a specific example is the stable $\mathbb{A}^1$-connectivity theorem.

Question 7

Question (Aravind Asok). *Fix an integer d. For certain d, Mohan Kumar constructs projective modules of rank $d-2$ which are stably free but not free over a smooth affine variety X of dimension d over an algebraically closed field. Is this known for all d? Is there a pattern?*

Question 8

Question (Jean Fasel). *Let X be a smooth affine k-variety of odd dimension d. Let P be a vector bundle of rank d. Does $c_d(P) \in CH^d(X)$ detect existence of a free summand of rank 1 for P?*

Question 9

Question (Kirsten Wickelgren). *Is there a Serre spectral sequence for Morel's $\mathrm{H}^{\mathbb{A}^1}(\_, \mathbb{Z})$?*

Question 10

Question (Anand Sawant). *When are the above questions well-defined over singular varieties? When have they been considered and what is their status?*

Question 11

Question (Paul Balmer). *Using operations like $\otimes, \wedge$ and taking cones, is there a construction so that one obtains the Koszul complex $K_\bullet(s)$ from the complex $\ldots 0 \to P \xrightarrow{s} A \to 0 \ldots$?*

Does this generalize to tensor triangulated categories?

Remark (Paul Balmer). Let $\mathbb{K}$ be a tensor triangulated category and let $X \in Ob(\mathbb{K})$. Suppose there is a map $X \xrightarrow{s} \mathbb{1}$. If the above construction works, then one can obtain a class $c(X) \in GW(\mathbb{K})$. This class should then define an obstruction class for X to split a free summand $\mathbb{1}$.

Remark (Madhav Nori). There is such a construction.

Question 12

Question (Anastasia Stavrova). *When G is a linear reduced/algebraic group, in a large number of cases, it is known on a case-by-case basis that $\Pi_0^{\mathbb{A}^1}(G)$ is abelian. Is this always true? Can this be obtained in a uniform way when it is true?*

Remark (Madhav Nori). This is not always true e.g. $G = SU(2, \mathbb{R})$. In fact, the same reasoning should show that $\Pi_0^{\mathbb{A}^1}(G)$ is not abelian for any anisotropic group G over $\mathbb{R}$.

Question 13

Question (Madhav Nori). *Given positive integers d, r and X a CW complex with $\dim X = d$ and classes $c_i \in \mathrm{H}^{2i}(X, \mathbb{Z})$ for $i = 1, 2, \ldots, r$, such that the image of c_i is 0 in $\mathrm{H}^{2i}(X, \dfrac{\mathbb{Z}}{m\mathbb{Z}})$ for all m, does there exist a complex vector bundle $\mathcal{E}$ with $\mathrm{rank}(\mathcal{E}) = r$ and $c_i(\mathcal{E}) = c_i$?*

Question 14

Question (Madhav Nori). *Let X be a d-dimensional smooth affine variety. Let $\alpha \in F^{d-1}K_0(X)$. Then does there exist a vector bundle $\mathcal{E}$ of rank $d-1$ such that $[\mathcal{E}] = \alpha$?*

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