Quantum curves, Hitchin systems, and the Eynard-Orantin theory

organized by
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Workshop Summary

The Scope of the Workshop. The workshop was planned in response to the recent explosive developments and interest around the notion of topological recursion and quantum curves. During the last few years, there have been numerous new research accomplishments. At the same time, more mysteries and questions have arisen. The purpose of the workshop is to bring the leading experts, some of whom are postdoctoral scholars and graduate students, and to seek answers of the fundamental questions of the field.

The Topics of Discussions and Talks at the Workshop. Reflecting the fact that the Workshop was focused on still an emerging young field of research, some of the morning talks were given by postdoctoral scholars and a graduate student, who had played the key role in their recent accomplishments.

- **Solution to the Remodeling Conjecture**
  The most sensational result in the area of research is the final solution to the *remodeling conjecture* due to Bohan Fang, Chiu-Chu Melissa Liu, and Zhengyu Zong.
  The first workshop/conference devoted to the topological recursion in the American Continent was held at AIM in June 2009, which attracted the main players in both mathematics and physics. In that occasion, mathematicians learned about the remodeling conjecture, formulated by physicists Vincent Bouchard, Albrecht Klemm, Marcos Mariño, and Sara Pasquetti. The conjecture predicts that certain generating functions of open Gromov-Witten invariants of any toric Calabi-Yau orbifold of dimension three should satisfy the topological recursion, base on the spectral curve which is determined by the mirror dual of the Calabi-Yau orbifold. The test case for the conjecture was a similar prediction for simple Hurwitz numbers formulated by Bouchard and Mariño. The Bouchard-Mariño conjecture was solved in 2009 by Bertrand Eynard, Motohico Mulase, and Brad Safnuk. Since then the remodeling conjecture has become one of the central problems in Gromov-Witten theory and topological recursion.
  The complete solution to the remodeling conjecture for the most general orbifold case was finally established by Fang, Liu, and Zong, after the pioneering work of Eynard and Nicolas Orantin. At the Workshop, Zong (graduate student at Columbia) gave an impressive talk describing this monumental work. The mathematical subtlety and difficulty of the problem required the team to work for 4 years to solve the conjecture. It is a significant achievement.
- **The New Developments in Topological Recursion**
  Eynard gave a survey of the state-or-the-art of the theory of topological recursion. His beautiful talk was an extended version of his Invited Address at the ICM 2014 in

Seoul, Korea. Since its inception of the theory in 2007 by Leonid Chekhov, Eynard, and Orantin, there have been many developments. At the same time, many new questions have been raised. New directions and current research efforts are also indicated in Eynard's talk. (The talk slides are available.)

Relations between the topological recursion and Painlevé equations are currently attracting the attention of many researchers. Vasilisa Shramchenko gave a survey of her recent work on the geometric realization of the Okamoto transform on some special solutions of the Painlevé VI, relating classical (Picard) solutions with the solutions discovered by Hitchin.

Some concrete examples of the topological recursion, exhibiting its power in counting problems, were given in the talk of Piotr Sułkowski, based on his joint work with Jϕrgen Andersen. The examples include computation of free energies for RNA molecules. Mathematically this amounts to computing certain generalizations of Catalan numbers.

When the spectral curve of the topological recursion is a disjoint union of open discs, the mathematical structure of the topological recursion becomes identical to that of Givental formalism of the Gromov-Witten potential for a semi-classical Frobenius manifold. This fundamental discovery was made by Petr Dunin-Barkowski, Orantin, Loek Spitz, and Sergey Shadrin.

Several generalizations of the original theory were also presented. A mathematical framework was presented by Gaëtan Borot, based on his joint work with Shadrin. Bouchard explained the generalization of the topological recursion for arbitrary ramification of the spectral curves, based on his joint paper with Eynard. Higher dimensional analogues were also discussed by Orantin.

The discovery of the topological recursion for a Hitchin's spectral curve of a Higgs bundle was reported by Olivia Dumitrescu (postdoctoral scholar at Hannover). This work, jointly obtained with Mulase, identifies the Eynard-Orantin spectral curves and Hitchin spectral curves, opening a door to a vast new uncharted territory of seemingly unrelated areas of mathematics. Her talk also indicated the generalization of the topological recursion for singular spectral curves.

- **Quantization**

  For the workshop participants, knowing the current mathematical theory of quantization was crucial for understanding the quantum curves. Andersen gave an inspiring survey of his recent work on quantization of Chern-Simons theory and Hitchin systems, based on his joint papers with Kenji Ueno. His talk has led the participants to recognize what would be necessary for constructing a mathematical theory of quantum curves.

  The classical WKB analysis was invented as a perturbative mechanism to construct a solution to a Schrödinger equation from the underlying classical mechanics. In recent years, modern theory of WKB analysis has been developed. Most recently, its relations to geometry of wall-crossing formulas for quantum invariants, such as Gromov-Witten invariants, and spectral networks arising from moduli theory of Higgs bundles, have been recognized. Kohei Iwaki (postdoctoral scholar at at Kyoto) gave a comprehensive introduction to the geometry of WKB analysis.

- **Quantum Curves**

Shadrin gave a survey of the current understanding of the quantum curves, based on the rigorous examples recently constructed by many mathematicians, including the most recent work on the Gromov-Witten invariants of ^{1}. The construction presented in this work, due to Dunin-Barkowski, Mulase, Paul Norbury, Alexandr Popolitov, and Shadrin, relies on a subtle property of representation theory of symmetric groups.

A totally different, more algebro-geometric construction of quantum curves based on birational geometry of ruled surfaces associated with Hitchin spectral curves, was presented by Dumitrescu. Relations to WKB analysis, non-Abelian Hodge correspondence, and Gromov-Witten invariants were also explained.

Research Activities Inspired by the AIM Workshop. Several concrete problems have been mentioned in the talks and afternoon discussions. Since then, many new collaborations have been launched in solving these problems. Papers are being produced from these collaborations as of now.

The workshop also facilitated previously non-existent interactions between different research communities, in particular, among the topological recursion, Hitchin systems, the WKB analysis, and quantization. Several participants are organizing, or giving plenary lectures in, international workshops throughout 2015 and 2016 on these new research interactions inspired at the AIM.

Fundamental Open Questions.

The participants exchanged the most recent accomplishments of the field in many different frontiers at the Workshop. A general feeling of satisfaction was felt, due to the significance of many of these achievements. The long anticipated remodeling conjecture was finally solved. The discovery of the relation to Higgs bundles and Hitchin systems was a total surprise. Still several fundamental questions are left unanswered.

- **Conceptual definition of the topological recursion.**
  The symmetric differential forms on a spectral curve are determined by the topological recursion as an inductive mechanism of definition. Yet what should be the straightforward definition of these differentials is still not well understood. In many examples, they are the Laplace transform of A-model quantum invariants. A general B-model definition of the differential forms is still missing.
- **Different constructions of quantum curves.**
  On one hand, the quantum curves associated with Hitchin spectral curves are constructed from the topological recursion and principal specialization by Dumitrescu and Mulase. On the other hand, many concrete quantum curves associated with various Hurwitz numbers, due to Mulase, Spitz, and Shadrin, and Bouchard, Daniel Hernández Serrano, Xiaojun Liu, and Mulase, and the above mentioned case of the Gromov-Witten theory of ^{1}, are constructed without topological recursion. The quantum curve itself for each of these cases is actually logically independent of the topological recursion. An application of the topological recursion similar for the Hitchin case to the Hurwitz and Gromov-Witten cases leads to a differential equation unrelated to the quantum curves.

  For the Hitchin spectral curve case, the mathematical definition of a quantum curve is understood through non-Abelian Hodge correspondence and Rees -modules. Yet these definitions do not cover the important examples associated with Hurwitz and Gromov-Witten cases, or those arising in knot theory through the AJ-conjecture.

A general definition of quantum curves is still missing.

- **The hidden relations between Hitchin spectral curves and the quantum invariants they compute.**

  When one starts with an A-model theory, one can often identify a spectral curve, and apply the topological recursion. In some cases one can also construct the quantum curve. In these cases, what information the quantum curve captures is known from the beginning.

  If one starts with a Hitchin spectral curve, applies the topological recursion of Dumitrescu-Mulase, and constructs the quantum curve by their method, then what does it calculate? A deep relation to Seiberg-Witten theory is manifest here, yet it was not investigated at the Workshop due to the lack of time.