# Quantum invariants and low-dimensional topology

organized by
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Workshop Summary

During the workshop we had daily working groups under the umbrella topics of skein modules of 3-manifolds, skein algebras, non semisimple TQFT's, and the Volume Conjecture. The specific questions evolved as they are partially resolved, and as the participants developed new inquiries. The problems and the progress made are reported by a person from each group below.

(1) **Study the colored Jones polynomials of hyperbolic knot complements with the same volume. Similarly, study the Turaev-Viro invariants of hyperbolic 3-manifolds with the same volume**—Effie Kalfagianni

The problem is motivated by the Volume conjectures asserting that certain asymptotics of the colored Jones polynomial (resp. Turaev-Viro invariants) should determine the volume of the knot complement (resp. the 3-manifold). Our group, which focussed mostly on the Turaev-Viro Invariants, had the following activities:

- One way to produce hyperbolic 3-manifolds with non-empty boundary with the same volume is to cut one 3-manifold along an essentially embedded thrice punctured 2-sphere $S_{0.3}$, and re-glue by a homeomorphism of $S_{0,3}$. Our group proved that this operation preserves the Turaev-Viro invariants and thus their asymptotics. Our proof relies on relations of the Turaev-Viro invariants to the $SU(2)$-Witten-Reshetikhin-Turaev (WRT) topological Quantum Field Theory and on the fact that the WRT 3-manifold invariants are preserved by certain operations called mutations.
- The "simplest" pair of once-cusped hyperbolic 3-manifolds that have the same volume but different Turaev-Viro invariants is the complement of the $5_2$ knot and the complement of the $(-2, 3, 7)$ pretzel knot. Numerical calculations verify that the Turaev-Viro invariants of the two manifolds are not the same. The asymptotic expansion conjecture for the Turaev-Viro invariants predicts that hyperbolic 3-manifolds with the same Turaev-Viro invariants should have the same Reideimester torsion (twisted by the adjoint action of the holonomy representation of the fundamental group of the manifold.)
We determined that the complements of $5_2$ and $(-2, 3, 7)$ pretzel knot have different torsions by computing their "1-loop" invariants (conjectually coincide with the torsion). Morally, this suggests that asymptotically the difference of the Turaev-Viro invariants is captured by the torsion.
- Hodgson, Meyerhoff, and Weeks have constructed infinitely many pairs of hyperbolic 3-manifolds with the same volume, Cherns-Simons invariants and cusp

volume. All the 3-manifolds discussed in their work are obtained by Dehn filling along a cusp of the Whitehead link. The "simplest" pair of such 3-manifolds is the complement of the knot $5_2$ and what is called "the sister of $5_2$". By finding explicit "ideal" triangulations of the two mainfolds one can see that they have the same Turaev-Virto invariants. Our group wondered if this holds for all the Hodgson-Meyerhoff-Weeks pairs. We are in the process of writing code for numerical calculations. At the same time, since the construction of the manifolds is quite explicit and each pair shares a common 2-fold cover we explore the possibility of constructing "similar" triangulations for each Hodgson- Meyerhoff-Weeks pair, that will allow to prove directly that each pair shares the same Turaev-Viro invariants.

(2) **Lift recursion relations of the colored Jones polynomial to the categorification**—Jennifer Brown

Our goal was categorifying the recursion relations of the colored Jones polynomial. Our starting point was an action of a quantum torus (namely, the $q$-Weyl algebra) whose kernel characterizes the recursive properties of functions such as the colored Jones. There's a topological interpretation in terms of gluing together skein modules of knot complements and solid tori, which we used as inspiration for manifesting a similar action on the colored Khovanov homology. In particular, we understood the categorified action as being in terms of the established recursive properties of categorified Jones Wenzl projectors together with a grading shift. We hoped to understand these as endofunctors, perhaps on the free cocompletion of the underlying ribbon category.

(3) **Compute skein modules of closed 3-manifolds**—Anup Poudel

We started by reviewing the $sl_2$ skein module of Lens spaces and the goal was to extend this for the case of $SL_3$ or higher skein modules. We successfully computed the elements of the $SL_3$ skein module of $RP^3$ by noting that under the handle-slide with 2-handles, the multicurves on an annulus, which in this case are given by two loops bounding the puncture and colored with two fundamental reps of $SL_3$ in two different orientations are linearly dependent. We discussed commutativity of the $sl_3$-skein algebra of the 3-punctured sphere and proved using the Frohman-Sikora coordinates for the $SL_3$ skein algebra that the skein algebra in this case is non-commutative. This was a surprising result since the associated $SL_2$ skein algebra of 3-puntured sphere is commutative.

(4) **Gluing stated skein algebras over a circle**—Benjamin Haioun

Our goal was to give a formula for gluing stated skein algebras over a circle. More precisely, given a (possibly non-connected) surface S with two chosen boundary circles $c_1$ and $c_2$, each of which have a single marked point, express the stated skein algebra of the glued surface $S/(c_1 = c_2)$ in terms of the stated skein algebra of $S$, and some of its boundary structure. We had in mind an expected formula, that the stated skein algebra of the glued surface is the $U_q(sl_2)$-invariants in the relative tensor product of the stated skein algebra over the two actions of the annulus (with a single marked point) actiong on each circle boundary component. We gave the construction of a cutting map from the glued surface to the cut surface. It took us a little longer to describe appropriately the two actions of the annulus, but we ended up with a concrete formula and a concrete map, which we conjecture is well-defined and an isomorphism.

(5) **Compare definitions of stated skein algebras at roots of unity**—Benjamin Haioun

There are multiple approaches to define algebras associtaed with $U_q(\mathfrak{sl}_2)$ and a surface with boundary which are known to coincide with stated skein algebras at $q$ generic, and we want to extend this comparison to $q$ a root of unity. We focused on stated skein algebras and internal skein algebras of Gunningham–Jordan–Safronov. The first ones are defined over inegral coefficients, and can be evaluated at a root of unity. The second ones are defined from some cocomplete ribbon categories, and there some choices appear. First one needs to choose a version of $U_q(\mathfrak{sl}_2)$ at roots of unity which is ribbon, there are at least the small quantum groups and Lustigz's divided powers. Then, one needs to choose either to work with locally finite modules over this Hopf algebra, or the free cocompletion of the category of finite dimensional modules. This second option seems unnatural in this context, but is more likely to agree with stated skein algebras. We could not make substantial progress towards actually comparing all these definitions though.

(6) **Which surface skein algebra modules are actually skein modules of 3-manifolds?**—Sam Panitch

Let $S(\Sigma)$ be the Kauffman bracket skein algebra of the surface $\Sigma$, and let $N$ be a module of $S(\Sigma)$. We asked whether we could realize $N$ as the skein module of a 3-manifold $M$, with the multiplication of $S(\Sigma)$ given by the inclusion of a slightly thickened $\Sigma$ into the boundary of $M$. We specialized to the case that $N$ was some quotient of a quantum torus, and our skein algebras defined by $\mathfrak{gl}(1)$ skein relations. We then spent some time studying the kernel of the inclusion map by considering how handle decompositions on a thickened surface affect the associated skein modules.

(7) **Comparison of Gukov-Manolescu two-variable series with Habiro invariant**—Sunghyuk Park

Using a certain completion of an integral form of quantum $\mathfrak{sl}_2$, Habiro showed in early 2000s that the colored Jones polynomials can be repackaged into a two-variable invariant that takes values in a certain completion of the Laurent polynomial ring in variables $q$ and $x$; the $n$-th colored Jones polynomial can be recovered by specializing $x$ to $q^n$. In 2019, Gukov and Manolescu predicted the existence of another knot invariant, $F_K$, taking the form of a power series in $x$ and $q$, from which colored Jones polynomials can be recovered. The invariant $F_K$ has since been rigorously defined for any closure of a homogeneous braid. While the variable $x$ morally plays the role of $q^n$ in both Habiro and Gukov-Manolescu invariants, the two invariants take very different forms, so our group discussed if they might be "the same" in some sense. We didn't come to any concrete conclusion, but one idea that came up was to think of them as distributions on the product of two unit circles.

(8) **The skein module of a 3-manifold is always non-trivial**—Giulio Belletti

The group worked on the problem of whether the Kauffman bracket skein module (with $\mathbb{Q}(A)$ coefficients) has to be non trivial for a closed manifold. This is known to be true for rational homology spheres; we tried to extend the techniques used to fit the general case. The first step in the proof is using the Gilmer-Masbaum momentum map; this allows us to reduce the problem to proving that any 3-manifold has non-zero Reshetikhin-Turaev invariants for infinitely many roots of unity. At this point there is a major difficulty: while it is easy to compute RT invariants for low levels (and show that they are sufficiently non-zero), these only cover a finite number of roots of unity. In the rational homology sphere case, this hurdle is handled by using the Habiro invariant, which is an invariant of the manifold that contains all the RT invariants; because of its algebraic properties (akin to those of analytic functions), if the Habiro invariant is non-zero at a root of unity, it must be non-zero at infinitely many. The aim of the group then became to try to extend the Habiro invariant to the general case; a straightforward implementation seems impossible but a (perhaps much) weakened version of the invariant could be achievable. If this new version could retain some of the continuity properties of the original Habiro invariant, there is a great likelihood that the initial problem could be solved in the affirmative.

(9) **Compute non-semisimple invariants for families of 3-manifolds and look at asymptotics**—Sanjay Kumar

We investigated the quantum invariants arising from nonsemisimple TQFT's at small level sets. In the case of the small quantum group, we computed that the admissible skein module of the sphere is 1-dimensional. Additionally, we demonstrated that the invariants from the TQFT are trivial for the L(p,1)-lens space. We also considered the TQFT associated with the quantum double of the Borel subalgebra with the goal of computing the corresponding quantum representations for torus bundles. In this example, we took the first steps of the calculation by verifying the dimension of the hom spaces between indecomposable representations.

(10) **Compare definitions of $SL_n$ skein modules and algebras**—Francis Bonahon

The goal of the group was to compare several versions of the $SL_n$-skein module of a 3-manifold, or the $SL_n$-skein algebra of a surface. We ended up focusing the discussion on two graphical versions, one due to Kuperberg-Sikora (KS) and the other to Cautis-Kamnitzer-Morrison (CKM). Poudel recently constructed an isomorphism between these two viewpoints, under the hypothesis that the quantum parameter $q$ is not a root of unity of order at most the dimension $n$. The case when $q$ is a root of unity is of special interest in particular because of the existence, in this case, of unexpected central elements in the skein algebra of a surface and of transparent elements in the skein module of a 3-manifold. The discussion revealed many subtleties. For instance, the Poudel homomorphism from the KS-skein algebra to the CKM-skein algebra is always defined but, when $q$ is a small root of unity, some KS-skeins are sent to 0 in the CKM-skein algebra without being obviously 0 in the KS-skein algebra.