Random walks beyond hyperbolic groups

organized by
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The goal of this AIM workshop was to extend results on random walks known in the Euclidean or Gromov hyperbolic case, to the more general class of groups which have (non-proper) actions on (non-proper) Gromov hyperbolic spaces. Our aim was to bring together researchers who have expertise in coarse geometry, geometric group theory and random walks, to extend recent results and to further develop new tools and directions. These goals were certainly fulfilled through this workshop.

Schedule of the week

As usual for AIM workshops, the program of every morning consisted of two talks, ranging from introductory talks to expositions of more recent developments that provided some of the motivation for problems that were posed for the working groups. The speakers and topics of the talks were the following:

- Vadim Kaimanovich: *An introduction to the Poisson boundary*
- Sebastien Gouëzel: *Exponential decay for random walks without moment conditions*
- François Ledrappier: *Dimension of harmonic measures*
- Inhyeok Choi: *Limit theorems for random walks on weakly hyperbolic groups*
- Joshua Frisch: *Poisson boundaries of ICC groups*
- Ilya Gekhtman: *The Martin boundary covers the Floyd boundary*
- Omer Angel: *Amenability of automaton groups*
- Alessandro Sisto: *Quasi-isometries and Markov chains*
- Kasra Rafi: *The sublinearly Morse boundary*
- Talia Fernós: *Poisson boundaries of CAT(0) groups*

On Monday afternoon, we collected a large list of open questions building off of foundational work in this field, of hugely varying difficulty, ranging from long-standing open problems, to problems which were felt to be fairly accessible. For the rest of the afternoons during the week of the workshop, the participants worked in groups on these problems. There were more problems than groups, and the collection of problems being worked on varied from

day to day. Furthermore, the groups were very dynamic, with participants changing their groups quite a bit throughout the week, and some groups coalescing into larger ones.

Main topics of discussion

In this section we give a brief overview of the main problems which were focussed on during the group discussions.

Poisson boundary of OOS groups..

Ol'shanksii, Osin, and Sapir [OOS] constructed examples of groups, known as *lacunary hyperbolic groups*, whose asymptotic cones are $\mathbb{R}$-trees, and used them to provide several counterexamples to old conjectures.

The initial question, proposed by M. Hull, was whether these groups may be possible sources of counterexamples to the *stability conjecture* of Kaimanovich [Ka]: for finitely supported measures, the triviality of the Poisson boundary only depends on the group, and is independent of the particular choice of a finitely supported measure.

Thus, the participants focused on understanding the Poisson boundary of such group, and whether it is trivial or not. The discussion seems to imply that such a boundary would always be non-trivial. A subset of the participants were able to find a sketch of the proof, and are planning to keep working on it and possibly write it down.

Random divergence..
The group discussed various notions of "random" divergence, generalizing the notion of divergence from geometry group theory to a random setting. The original notion of divergence in a group was developed by Drutu-Mozes-Sapir [DMS], and captures how long a path in the Cayley graph of the group which avoids a ball of radius R must be. The question, proposed by A. Sisto, was to define a notion of divergence between two random points, constructed by taking two independent random walks, and then use it as an invariant to distinguish different groups up to quasi-isometries. The group investigated this question, and realized that in most situations the new version of random divergence tends to behave exactly like the usual divergence. Hence, the question was not pursued further.

Singularity of measure for Cannon-Thurston maps and geometrically infinite Kleinian groups..

This group sketched out a possible approach for the simplest case of the original Cannon-Thurston map, arising from a surface in a hyperbolic 3-manifold which is a fiber of a fibration. This gives a (surjective) map from the boundary of the universal cover of the surface (S^1) to the boundary of the universal cover of the manifold (S^2). There are four natural measures arising from this construction on S^2, the standard Lebesgue measure for S^2, the push forward of Lebesgue measure on S^1 by the Cannon-Thurston map, and the two correpsonding hitting measures from random walks on the surface group and the 3-manifold group. The first two of these measures should be mutually singular, and it should be possible to show this by considering the corresponding quasi-geodesics in the singular solv metric, which is quasi-isometric to the hyperbolic metric on the universal cover of the 3-manifold. The group intends to attempt to write up the details of this, and see if it extends to any of the hitting measures, or any of the more general Cannon-Thurston map constructions.

Generic elements in Cayley graphs..

This group reviewed Choi's recent work [Ch] on generic elements in surface mapping class groups, and some of the earlier literature. For mapping class groups, this is a long-standing open problem ([Fa], Conjecture 3.15) which probably requires new techniques, and no further progress was made, though the groups considered looking for simpler groups which might be easier to work with.

Dimension of harmonic measure on Moduli space..

The question discussed in this group, proposed by G. Tiozzo, concerns the dimension of harmonic measures for random walks on cocompact Fuchsian groups. Let μ be a fixed, finitely supported measure on a closed surface group Γ, and let us look at the space of all representations $\rho : \Gamma \to SL_2(\mathbb{R})$. As is known ([Le], [Ta]), the dimension of harmonic measure ν for the random walk driven by μ is given by

$$\text{H.dim } \nu = \frac{h}{\ell}$$

where h is the entropy and ℓ is the drift. It is conjectured that such dimension is always strictly less than 1 for all cocompact representations ([KP], [KT]).

The group set out to prove the following related statement. Let ℓ_ρ be the drift of the corresponding random walk driven by μ on $\rho(\Gamma)$: then the drift ℓ_ρ of the random walk tends to infinity as ρ tends to infinity in Teichmüller space. This has as its consequence that the dimension of harmonic measure tends to 0 as ρ tends to infinity.

The group analyzed several ideas of proofs, and then they settled on using the technique of studying the limiting representations given by having ρ tend to infinity, and then using the work of Morgan-Shalen [MS] that studies these degenerations as actions of Γ on certain trees. A joint paper on this project is currently being written.

Sublinear Morse boundary includes in the Martin boundary, extensions of the Ancona inequality..

As proposed by I. Gekhtman, the group discussed the relation between the recently defined sublinearly Morse boundary [QRT] and the Martin boundary. In particular, they aimed to prove that there is a natural inclusion of the sublinearly Morse boundary into the Martin boundary, similarly as to what is known for the Gromov boundary.

The group realized that the original goal was too ambitious, but they identified a smaller subset, namely the set of *frequently Morse* geodesics, and believes that this subset should be generic for the random walk, and it should embed into the Martin boundary.

The group observed that in order for that to be true, it is necessary to prove an extension of the Ancona inequality [An], replacing the fixed error bounds with sublinear bounds.

The group was able to make substantial progress on this topic, and is planning to talk again and keep working on it.

Poisson boundary for free semigroups..

The group focused on the problem of identifying the Poisson boundary for random walks on free groups and free semigroups. If the measures have finite entropy and finite

logarithmic moment, the Poisson boundaries for such groups are known for a long time, by work of Kaimanovich. However, the question for general measures is still open.

The group discussed what is currently the most general result for free semigroups [FT], and tried to generalize its approach, as well as formulating some equivalent formulations of the main conjecture.

Then, they focused on the simpler case of the free group under the finite entropy assumption. This is still open in this generality. The group tried to use Gouëzel's pivoting technique [Go] to solve this, and they were able to sketch a proof. At the moment, the group is trying to pin down the details of the proof, and to write the paper.

Linear progress for Markov chains..

Initially, the problem (proposed by A. Sisto) was to show a generalization of the linear progress result from random walks to Markov chains on weakly hyperbolic groups, so that it is quasi-isometry invariant. After a while, the group realized that the goals for this project were quite closely related to Problem , hence they merged with that group.

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