Rationality problems in algebraic geometry

organized by
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Workshop Summary

Overview

This workshop was devoted to topics related to the rationality of algebraic varieties. The focus was on the following areas:

(1) Techniques for proving that varieties are not rational, especially via degenerations and unramified cohomology.
(2) Techniques for proving varieties are rational or unirational.
(3) Relations between derived categories and rationality.

The participants in the workshop had expertise in different areas — some primarily had background in "classical" topics like birational geometry and Hodge theory, while others were more familiar with derived categories. Thus the first goal of the workshop was to the familiarize the participants with the basics of the above areas. The second goal was to identify promising future directions in these areas.

The workshop followed the AIM format. The mornings were dedicated to talks. These talks were of an introductory nature for the first couple days, and then shifted towards more current research towards the end of the week. A highlight was Ludmil Katzarkov's talk on his ongoing work with Kontsevich and Pantev, where they construct new birational invariants that are strong enough to prove the irrationality of cubic fourfolds, which is one of the biggest outstanding problems in the field.

Monday afternoon consisted of a productive open problem session. In the afternoons for the rest of the week, the participants worked in groups on problems. The outcomes of these working groups are summarized below.

Working group projects

Unirationality of hypersurfaces.
This group worked on improving the range of pairs of integers (d, n) for which a degree d smooth complex hypersurface $X \subset \mathbf{P}^n$ is provably unirational. Recently, Beheshti and Riedl (improving work of Harris, Mazur, and Pandharipande) showed that $2^{d!} \leq n$ suffices to ensure unirationality. The method of proof is based on an induction on d, so an improvement to the bound for some small d leads to a better bound in general.

The group focused on proving unirationality of degree 6 hypersurfaces $X \subset \mathbf{P}^n$. By some classical geometric arguments, they relate this to proving unirationality of complete intersections of type $(2, 3, 4)$ and $(2, 3)$ over non-closed fields, where progress can be made by building on a result due to Enriques. Altogether, this gives a promising strategy for significantly improving the range of dimensions for which unirationality is known for (general) degree 6 hypersurfaces. The group plans to continue working on this project.

Integral Hodge conjecture.
While the integral Hodge conjecture (IHC) holds for rationally connected threefolds by a result of Voisin, there are six-dimensional counterexamples by Colliot-Thélène and Ojanguren, as well as four-dimensional ones by Schreieder. The counterexamples are established via a result of Voisin and Colliot-Thélène, which relates the failure of the IHC for codimension two cycles to the nontriviality of the third unramified cohomology with $\mathbb{Q}/\mathbb{Z}$-coefficients. Given the nature of this proof, it remains unclear whether the non-algebraic Hodge classes produced this way are torsion or non-torsion.

This group studied the aforementioned examples of Colliot-Thélène, Ojanguren, and Schreieder, and tried to decide whether the non-algebraic Hodge class is torsion or not. The group found that this question can be answered via an obstruction class in the Chow group of codimension two cycles and they tried several ways of computing this class in the given examples. However, a full answer to the original question remains open for the time being.

Derived categories of quartic double fivefolds.
This group studied the relation between the rationality of quartic double fivefolds and their derived categories. For such a fivefold X, there is a Kuznetsov component $\mathcal{K}u(X) \subset \mathrm{D^b}(X)$ which is a 3-Calabi–Yau category. Kuznetsov's rationality for conjecture for cubic fourfolds has an analogue in this setting: a smooth X is rational if and only if $\mathcal{K}u(X)$ is equivalent to the derived category of a Calabi–Yau threefold. In fact, $\mathcal{K}u(X)$ is never equivalent to the derived category of a Calabi–Yau threefold, so the conjecture predicts that X is irrational. This is currently unknown.

The group instead studied certain singular degenerations of X which can be proved to be rational. They showed that in this setting, there is a crepant resolution of the category $\mathcal{K}u(X)$ by the derived category of a Calabi–Yau threefold. This verifies a degenerated version of the above conjecture. The group is currently writing a paper on this result.

Stable irrationality of complete intersection fourfolds.
This group considered the stable rationality problem for complete intersections $X \subset \mathbf{P}^6$ of a quadric and a cubic over the complex numbers. These are Fano fourfolds that, together with cubic fourfolds, constitute the last remaining open case in the stable rationality problem for Fano complete intersection fourfolds. The rest of the cases can be covered by existing applications of the degeneration methods for 0-cycles.

The group tried to apply the degeneration method by finding an appropriate singular X whose universal triviality of CH_0 is obstructed by a nontrivial unramified Brauer class. In their first attempt, they considered those X that contain a plane; projecting from this plane produces a model that has a conic bundle structure over $\mathbf{P}^3$. They computed a nice presentation for the discriminant of this conic bundle (which has degree 7), but were unable to find reasonable conditions under which it became sufficiently reducible so as to apply the existing techniques.

In their second attempt, they considered those X that contain a linear subspace of dimension 3; projecting from this linear space produces a model that has a quadric surface bundle structure over $\mathbf{P}^2$ with octic discriminant. The hope was to find a member of this family that is birational to the example discovered by Hassett, Pirutka, and Tschinkel. The group was able to computationally find members where the discriminant profile matched, but the residue data always seemed to collapse. Since the workshop, the group has been considering other ways to find a special member with the desired properties.