Rational and integral points on higher-dimensional varieties

organized by
Tim Browning and Yuri Tschinkel

Workshop Summary

The focus of this workshop was on arithmetic and geometric problems arising in the study of rational and integral points on higher-dimensional algebraic varieties. This is a rapidly growing area at the interface between classical analytic number theory and algebraic geometry.

Here is a short list of key issues and open problems:

(1) Brauer–Manin obstructions to the Hasse principle and weak approximation for rational and integral points;
(2) density of points on higher-dimensional varieties;
(3) solvability of random equations;
(4) equidistribution;
(5) applications of additive combinatorics in geometric settings and generalisations of the circle method.

Among the most spectacular advances is recent work by Bhargava and his coauthors establishing first nontrivial statistics on ranks of elliptic curves in families. This topic was introduced to the workshop participants by Bhargava and played an important role in afternoon discussions. Indeed, several working groups thought about applications of these results to solvability of diagonal cubic equations of the type

$$ax^3 + by^3 + cz^3 + dt^3 = 0,$$

and related equations.

In the second talk of the meeting, Peyre proposed a variant of Manin's conjecture in which one counts rational points of bounded height on varieties which are sufficiently "free". Preliminary investigations suggest that this point of view will permit one to circumvent the contribution from rational points lying on so-called "accumulating subvarieties", without having to identify the varieties ab initio.

Another recent result, a construction of infinitely many Kummer type K3 surfaces over the rationals satisfying the Hasse principle, by V. Pal, was explained in detail by Alexei Skorobogatov. Again, advances in the theory of elliptic curves lead to new, highly nontrivial results for surfaces.

Arne Smeets talked about his very interesting new counterexample to the uniqueness of Brauer–Manin obstructions. Tony Várilly-Alvarado reported on joint work with McKinnie, Sawon and Tanimoto introducing a completely general approach to the construction of Azumaya algebras of order p on K3 surfaces of geometric Picard rank 1. This explicit presentation of p-torsion elements of the transcendental Brauer group may allow one, at least in principle, to compute effectively the set $X(\mathbb{A}_k)^{\mathrm{Br}}$ of Brauer–Manin unobstructed adèles, for every K3 surface X over a number field k.

In another talk, Skorobogatov formulated a striking conjecture, predicting the density of $X(k)$ in $X(\mathbb{A}_k)^{\mathrm{Br}}$ for K3 surfaces. Combined with his results with Zarhin on the finiteness of $\mathrm{Br}(X)/\mathrm{Br}(k)$, this conjecture would also imply potential density of rational points on K3 surfaces over number fields.

Wittenberg spoke about his recent joint work with Harpaz, in which an interesting new generalisation of Schinzel's hypothesis is proposed. If true it allows very general conclusions to be drawn about rational points on varieties. The underlying tool is a new non-abelian fibration method.

The talk by Lilian Matthiesen laid out very clearly the main steps in her recent joint work with Browning and Skorobogatov establishing the uniqueness of the Brauer–Manin obstruction for conic bundles over the rationals. The problem has been open for more than 30 years, and this breakthrough, which combined analytic ideas from additive combinatorics (Gowers' norms) with the theory of universal torsors, was one of our motivations to hold the workshop. Indeed, the flow of ideas and "technology" between analytic number theory and higher-dimensional geometry is clearly accelerating, and our goal is to stimulate the emergence and development of similar success stories.

Here are some sample questions from analytic number theory to geometry, that may not have occurred without our dialog:

— What are the geometric issues in studying diagonal quadrics over $\mathbb{Q}(t)$?
— Do varieties within the range of the circle method have trivial fundamental groups?
— Is there a sequence of varieties $X_d \subset \mathbb{P}^n$ of dimension d over $\mathbb{Q}$ such that

$$\frac{1}{3} < \frac{d}{\deg(X_d)} < 2$$

and $\mathrm{Br}(X_d)/\mathrm{Br}(\mathbb{Q}) \neq 0$?

In turn, here are some requests to number theory:

— Extend the circle method to O_S-integral points, where S includes some primes.
— Extend the point counting on universal torsors of singular Del Pezzo surfaces to allow for integrality conditions with respect to some of the exceptional curves; prove an extension of Manin's conjecture to the context of integral points.
— Explore analytic approaches to potential density of rational points on geometrically non-rational conic bundles over $\mathbb{P}^2$.