Algebraic systems with only real solutions

organized by
Evgeny Mukhin, Natasha Rozhkovskaya, and Vitaly Tarasov

Workshop Summary

The workshop was devoted to a new phenomenon recently discovered in algebraic geometry, known as the B. and M. Shapiro conjecture. Roughly speaking, the conjecture asserts that various problems of the Schubert calculus related to real points on normal rational curves have real solutions. The B. and M. Shapiro conjecture for the Grassmannians has been proved by A. Eremenko and A. Gabrielov in the case of planes, and by E. Mukhin, V. Tarasov and A. Varchenko in the general case. The conjecture for the Grassmannians can be reformulated in the following way: if all roots of the Wronskian of several polynomials are real, then the space spanned by these polynomials has a basis consisting of polynomials with real coefficients. The original B. and M. Shapiro conjecture the flag manifolds is known to be false. However, F. Sottile suggested its modification, known as the Monotone conjecture, and confirmed it with collaborators by the extensive numerical calculations.

There are different approaches to the problem using algebraic geometry, complex analysis, and integrable systems. The purpose of the workshop was to bring people with different approaches together in an attempt to compile the list of the questions and ideas related to the area.

The following events took place during the workshop.

F. Sottile presented an overview of the state of art of the problem. He explained that in many cases the B. and M. Shapiro conjecture fails in an "interesting way." This gives rise to the Monotone conjecture as well as some other conjectures, though with not so much supporting numerical evidence.

The talk by M. Shapiro explained the original motivation for the B. and M. Shapiro conjecture which comes from the concept of "disconjugate differential equations." This notion was a new subject for many participants and, therefore, it created a lot of discussions. A monic differential equation of order n with real rational coefficients is called disconjugate on a real interval if every solution of this equation has at most $n-1$ roots on the interval. During the workshop it was conjectured by A. Eremenko that all Fuchsian differential equations with only polynomial solutions and real singular points are disconjugate on intervals between the neighboring singular points. He explained that this is crucial for his approach to the proof of the so-called Secant conjecture.

The Secant conjecture was in the center of many discussions. It claims that under certain separation conditions, the point on the Grassmannian which meets the expected number of the secant spaces through real points of the normal rational curve is real. A. Eremenko conjectured that all solutions of the secant Schubert problem corresponding to real points also solve the standard Schubert problem related to real points on the rational normal curve. This surprising conjecture was tested and confirmed numerically in several examples by F. Sottile.

The Eremenko-Gabrielov proof of the B. and M. Shapiro conjecture is based on the concept of nets. Nets are certain combinatorial invariants which are computed from the solutions of the Schubert problem for the Grassmannian of planes. A net is the preimage of the real axis under the rational function which is the ratio of polynomials spanning the plane (a point of the Grassmannian). During workshop there was a working group led by A. Gabrielov, R. Goldin and J. Timoczko, trying to construct similar objects, called multinets, in the case of the Grassmannian of three-dimensional spaces. In that case, rational functions are replaced by plane curves. Multinets were constructed for quintics and some progress was made for sextics.

The B. and M. Shapiro conjecture extends naturally to the case of spaces of quasi-exponential functions (products of polynomials and exponentials). M. Yakimov in his lecture explained how the generalized conjecture is related to the bispectrality property and Darboux transformations. For the rank one case of the bispectrality problem, the generalized B. and M. Shapiro conjecture is equivalent to the fact that the reality of the Darboux transformation follows from the reality of its support. A natural question is if this property extends to higher rank cases of the bispectrality problem. This extends the B. and M. Shapiro conjecture to some classes special functions, for example to generalized Airy and Bessel functions.

A. Eremenko explained the relation of the Wronski map to problems in Control theory, where it provides the first counterexample to the pole placement problem. Inspired by this talk, F. Sottile tested on a computer the stabilizability property and found a counterexample to this old and important problem right during the workshop.

F. Sottile conjectured that the discriminant of the Wronski map is always a sum of squares in terms of the roots of the Wronskian. E. Mukhin pointed out that in the simplest situation it implies the following conjecture: for any polynomial $w(x) = \prod_{i=1}^{n}(x - z_i)$, the discriminant of the derivative $w'(x)$ is the sum of squares of products of the differences $z_i - z_j$. It was tested to be true for $n = 3$ and $n = 4$, but no immediate proof or counterexample was found for such a classically looking statement.

K. Purbhoo explained the combinatorics related to asymptotic analysis of the B. and M. Shapiro conjecture for the Grassmannians. It is given in terms of jeu de taquin for the Young tableaux. A discussion started during one of the dinners led to realization that if one takes the points on the rational normal curve which correspond to complex numbers lying on a circle in the complex plane with real center then the number of real solutions to the corresponding Schubert problem can be computed using the jeu de taquin.

V. Tarasov gave several tutorials on the Bethe ansatz proof of the B. and M. Shapiro conjecture. It is unclear for the moment if a similar proof can be worked out for the case of the Monotone conjecture.

N. Hein and Z. Teitler computed multiple data related to the non-Monotone case of the B. and M. Shapiro type problems for the partial flag variety. It is plausible that under some conditions all solutions of such problems are real.

There were other topics discussed at the meeting in connection with the main subject. They included, in particular, lower bounds for the number of real solutions (M. Azar, A. Gabrielov), geometry and topology of intersections of Schubert varieties (B. Shapiro), relation to M-varieties (B. and M. Shapiro), and connection to Kazhdan-Lusztig polynomials (A. Yong).