Arithmetic reflection groups and crystallographic packings

organized by
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Workshop Summary

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This workshop was devoted to the complete classification of maximal hyperbolic arithmetic reflection groups and related questions. About 15 years ago, the number of such groups was shown to be finite, leaving open the possibility of fully classifying them in a way that makes them accessible for other applications, e.g., to the study of crystallographic sphere packings. Recent advances, both theoretical and algorithmic, made it plausible that with sufficient collaborative effort by experts in diverse areas from geometry/topology, dynamics, arithmetic groups, and number theory, this program could be completed. The goal of the workshop was to foster collaborations between people working in these different areas.

Due to various travel restrictions, there were seven in person attendees. Because of the small size, the workshop had many features in common with a SQuaRE. The talks were informal, and often also impromptu. We were also able to leverage technological advances, to hear virtual lectures from and hold discussions with invited participants who were unable to travel.

The speakers were:

[itemsep=.1in,leftmargin=.7in]Daniel Allcock Rudolph Scharlau (via Zoom), and Mathieu Dutour Anna Felikson, and Nikolay Bogachev (via Zoom) Alex Kontorovich, Arthur Baragar, and Ian Whitehead

After the first talk on Monday morning, there was a moderated problem session, at which open problems were formulated and discussed. More problems were added to the list as the week progressed. The final list is as follows:

[itemsep=.1in,label=0.]Can we extend Bogachev's method (improving on Nikulin's work) on the outermost edge beyond restrictions on root norms? This group looked for variations on this theme, trying to find something analogous to Allcock's short pairs and close pairs for H^3. In which dimensions are there arithmetic reflective lattices over $\mathbb{Q}$ with an isolated root?

(2)
- They are known to not exist in dimension 20 or higher (including dimension 21, despite the existence of the Borcherds lattice in this dimension).
- The missing dimensions are 16 and 17. (See recent work of Bogachev-Kolpakov-Kontorovich.)
- Answering this question would also determine exactly what are the dimensions where there are superintegral crystallographic sphere packings.

(3) Do there exist right-angled Coxeter groups in H^n with $n > 8$?

(4) Make Coxeter's classification of finite and affine reflection groups formally computer verifiable (say, in the Lean interactive theorem prover).

(5) Given a Coxeter group, determine whether it is a proper subgroup of a Coxeter group in the same dimension.
(6) Is there a (reasonable) way to bound the volume of a Weyl chamber of a reflective lattice under some assumptions (say, strongly-squarefree)?
(7) Implement Selberg's Algorithm in higher rank to determine whether a Zariski-dense subgroup of an arithmetic group, given by some generators, is a lattice or "thin".
(8) Must integral sphere packings come from quasi-arithmetic lattices? Does there exist a properly integral packing whose supergroup is not even quasi-arithmetic?
(9) Find new constructions of quasi-arithmetic reflection groups.
(10) Come up with a version of Vinberg's Algorithm for groups generated by finite order symmetries.
   - Is there a way to compare them? Like Coxeter diagrams?
   - Is there a bound on their dimension?
(11) Is there a level 2 root system with a large level 1 π-subsystem of the same rank? More broadly, is there an automorphically correctable root system contained in a crystallographic diagram?
(12) Is the λ-length formula the same as Descartes's circle theorem?
(13) Which polyhedra are arithmetic? Kontorovich-Nakamura define a (combinatorial type of a convexly-realizable) polyhedron to be *arithmetic* if: when geometrized (by the Koebe-Andreev-Thurston theorem) to have a midsphere, the resulting hyperbolic 3-fold obtained from reflection in the walls of the polyhedron and its dual, is arithmetic. This will necessarily be of simplest type, non-uniform, over $\mathbb{Q}$. (Such also generate superintegral crystallographic circle packings "modeled" on the given polyhedron.)
   - What is the full set of seed arithmetic polyhedra?
   - Can you tell if a packing is polyhedral?
(14) What are the lattices preserving $\begin{pmatrix} 1 & & -1 \\ & \ddots & \\ -1 & & 1 \end{pmatrix}$? In which dimensions is it reflective?

Over the course of the week, progress was made on questions 1, 4, 7, 13 and, 14 Additionally, progress was reported by Scharlau and Kirschmer on a prerequisite for answering question 2. What follows is a summary of what was done on each of these questions.

Question 1.
The method of the outermost edge gives a finite list of candidate forms by bounding the inner product of the two endcaps. However, it only works if you have some control over the norms of the roots. An outermost edge has four incident faces, and as long as they can't be split into two subsets where all the faces in one meet all the faces in the other at right angles, you have the control you need. However, a lot of possible configurations have a lot of right angles, so we need to be able to handle such cases.

Bogachev handled them in his classification by restricting to lattices with a particular fixed list of root norms. But if you want a list of all possible candidate lattices, you can't assume such a restriction. Inspired by Allcock's short edge/short pair/close pair features of hyperbolic polygons, we thought about ways to add more faces systematically to deal with all possible cases. Though we did not come up with a full analogy to the 2-dimensional case, we did deal with some specific 3-dimensional cases, and we also talked about a way to think about shortness, and the level of shortness, of a facet in any number of dimensions.

Question 2.
Scharlau and collaborators (Walhorn, Turkalj, Kirschmer) should "eventually" have a complete list of arithmetic lattices over $\mathbb{Q}$ in these dimensions, and then hopefully the question can be answered definitively (though this is not immediate, and more work may remain). In the meantime, we tried to jump the gun and construct in 17-dimensional hyperbolic space many potential lattices to check from genus theory, and ran Vinberg's/Allcock's algorithms on them. A "new" lattice having 48 roots was discovered in dimension 17; Allcock identified this as the $D_4 \times D_4$ face of the Conway lattice in $25 + 1$ dimensions. The known 960-rooted Borcherds lattice in dimension 17 was not (yet) found; the computations are ongoing. Another potential source to investigate is involution centralizers in $II_{17,1}$.

Question 4.
This group made the first steps towards the classification in the Lean theorem prover, but much more work remains.

Question 7.
We attempted an attack on the explicit problem of a pair of 3×3 matrices given by Long, Reid, and Thistlethwaite; it known that the group generated by these is either all of $SL_3\mathbb{Z}$ (which we should have been able to determine by now), or is thin. To prove the latter, we tried implementing Selberg's fundamental domain algorithm in symmetric spaces. Starting with a base point in the positive-definite cone, we found very many bounding walls, but all of its vertices were outside the cone. In the case of Vinberg's algorithm, one would then look for a symmetry of the portion of the fundamental polygon thus found, lying in the ambient integer orthogonal group, having infinite order. But in the case of this group, there is no "ambient" group in which to determine infinite order symmetries. Moreover, the walls thus far found are not guaranteed (unlike Vinberg) to be the "nearest" ones to the base point, and may become obsolete with more computation. One needs some kind of new version of Poincare's Fundamental Polyhedron Theorem in this setting. Another idea (not yet pursued) is to try to determine/compute the normalizer of this group.

Question 13.
Any such is commensurable with some reflective extended Bianchi group, or better yet, a subgroup of the Scharlau-Walhorn classification of the maximal arithmetic reflective lattices over $\mathbb{Q}$ of signature $(3, 1)$. We tried to carefully unfold these domains, seeing if we could rule out the possibility of some subgroup decomposing into a polyhedral cluster/cocluster pair. Conjecture (Kontorovich-Nakamura): Arithmetic polyhedra are only commensurable with $PSL_2\mathbb{Z}[i\sqrt{d}]$ with $d = 1$ (tetrahedron, and its "growths"), 2 (square-pyramid), or 6 (cuboctahedron). Here is one possibility for how one might prove that there is no cover of a Bianchi group having only ideal and right angles, and separated into a cluster/cocluster pair of such: "Lemma" (whose proof needs to be checked): Given a tile, if it has two finite corners having no ideal "completion", then there cannot be such a cover. If we can demonstrate

this condition for all other maximal reflective Bianchi groups, the conjecture will be proved! (Work is ongoing.)

Question 14.
It is reflective up to dimension 8, which means that it gives rise to sphere packings up to dimension 6.