Research experiences for undergraduate faculty

organized by
Leslie Hogben, Roselyn Williams, and Ulrica Wilson

Workshop Summary

This was the third in a series of workshops for undergraduate faculty. The goals of REUF are:

- To provide faculty participants the opportunity to have a research experience investigating open questions in the mathematical sciences.
- To prepare participants to engage in research with undergraduate students at their home institutions, in some cases using the problems presented.
- To rekindle or further fuel a love of doing original mathematics.
- To involve some participants in long-term research collaborations.
- To support faculty participants in their efforts to engage undergraduates in research and to continue their own new research collaborations formed through the REUF workshop.

The four mathematical leaders, Ruth Haas of Smith College, Loek Helminck of North Carolina State University, Linda Patton of California Polytechnic State University -San Luis Obispo, and Ilya Spitkowski, College of William and Mary, presented problems Monday morning and research groups formed Monday afternoon. The groups worked throughout the week and presented reports Friday afternoon (see Section . There were also whole group discussions of practical considerations of doing undergraduate research (see Section), the computer mathematics system *Sage*, and workshop continuation activities.

Group reports

Exponential graph domination.
L. Boos, M. Craddock, S. Dutta, R. Hass, A. Salerno, A. Shaqlaih, R. Taylor

A *graph* $G = (V, E)$ is a set of vertices V, together with a set of pairs of vertices called edges E. A **dominating set** S is a subset of V such that for every vertex x in V either 1) x is in S, or 2) x has some neighbor y which is in S. The **domination number** of a graph G, $\gamma(G)$, is the size of the smallest dominating set. There is a wealth of literature about graph domination and variations of this concept.

This REUF group studied a variation known as *domination with exponential decay.* Here, a vertex in the dominating set partly dominates all vertices in the graph. Formally, given a set $S \subset V$, define a function:

$$w_S(v) = \begin{cases} \sum 1/(2^{d(u,v)-1}) & \text{if } v \notin S, \\ 2, & \text{if } v \in S \end{cases}$$

where $d(u, v)$ is the length of a shortest (u, v) path that does not use any other vertices in S. The set S is an *exponential dominating set* for G if for all vertices $v \in V$, $w_S(v) \geq 1$.

The exponential dominating number, $\gamma_e(G)$ is the least number of vertices in an exponential dominating set.

The REUF group obtained results about exponential domination on grid graphs and generated a catalogue of related problems for further research both by faculty members and for undergraduates.

Structure of Symmetric k-Varieties.

K. K. A. Cunningham, T. J. Edgar, A. G. Helminck, B. F. Jones, H. Oh, R. Schwell, J. F. Vasquez

Let G be an algebraic group over a field k, and suppose that $\theta \in \mathrm{Aut}(G)$ such that $\theta^k = id$. We are initially interested in classifying such θ up to equivalence, where equivalence is given by conjugation by inner automorphisms, outer automorphisms or both depending on which makes the most sense.

We then define the following two sets

$$H = G^\theta = \{g \in G \mid \theta(g) = g\}$$
$$Q = \{g \in G \mid g = x\theta(x)^{-1} \text{ for some } x \in G\}$$

where H is the fixed-point subgroup of θ and Q is known as a symmetric k-variety. Symmetric k varieties play an important role in representation theory and harmonic analysis.

Both G and H act on Q by θ-twisted conjugation. That is for $g \in G$ and $q \in Q$ we have

$$g * q = gq\theta(g)^{-1}.$$

Since H is the set of fixed points, this action is simply conjugation when restricted to H. In fact, if θ is an involution $Q \cong G/H$ under the map $\tau : G \to Q$ given by $\tau(g) = g\theta(g)^{-1}$.

We are interested in classifying G and H orbits in Q and how the G-orbits decompose into H-orbits. In particular, if θ is an involution it is known that $H\backslash Q \cong H\backslash G/H$. These orbits and double cosets also play an important role in representation theory. We investigated these structures and questions for specific groups.

Symmetry of Numerical Range.

L. Deaett, R. Lafuente, J. Marin, E. Martin, L. Patton, K. Rasmussen, R. Yates

The numerical range of an $n \times n$ matrix A is the set defined by $W(A) = \{\langle Av, v\rangle \mid v \in \mathbb{C}^n, \|v\| = 1\}$. The numerical range of a matrix is always a convex set that contains the eigenvalues of the matrix.

In 1991, Li and Tsing showed (within more general results) that a collection of generalized numerical ranges of a matrix (one of which is the classical numerical range) all have n-fold symmetry about the origin if and only if the matrix is unitarily equivalent to a special block form. It was recently shown by Cal Poly REU students along with Spitkovsky and me that under certain natural conditions the classical numerical range of a 3×3 matrix has 3-fold symmetry about the origin if and only if the matrix is unitarily equivalent to the special form given by Li and Tsing; in addition, equivalent conditions for numerical range symmetry which are simpler to test with the matrix were provided.

The group of six faculty members working with me at AIM's REUF generalized some of the sufficient conditions for n-fold symmetry of the numerical range to $n \times n$ matrices and found a counterexample in the 4 by 4 case that shows Li and Tsing's block form is not necessary if just the classical numerical range is required to have four-fold symmetry about the origin. The group plans to continue work on the problem.

Problems related to Numerical Range.

E. Benander, P. Rault, K. Camenga, T. Senova, I. Spitkowski, R. Williams, M. Young

Recall that the numerical range of an $n \times n$ matrix A is defined as

$$W(A) = \{(Ax, x) \colon \|x\| = 1\}$$

and is a convex compact subset of the complex plane $\mathbb{C}$. For 2×2 matrices A, $W(A)$ is an elliptic disk, in the case of normal A degenerating into a line segment. The shape of $W(A)$ is also described completely in the case $n = 3$ but for higher dimensions less is known, and any new results in this direction are of interest.

The group was offered three problems related to numerical range descriptions of several classes of matrices, and actual work was done on two of them. The brief overview of these problems and the progress achieved is given below.

1. Numerical ranges of 4×4 doubly stochastic matrices.

A matrix A is *doubly stochastic* if its entries a_{ij} are non-negative, and

$$\sum_{j=1}^{n} a_{ij} = 1 \ (i = 1, \dots, n), \quad \sum_{i=1}^{n} a_{ij} = 1 \ (j = 1, \dots, n).$$

Such matrices have a normally splitting eigenvector corresponding to the eigenvalue 1, and therefore are unitarily similar to $(1) \oplus B$ for some $(n-1) \times (n-1)$ matrices B. Consequently, $W(A)$ is the convex hull of $W(B)$ and the point 1. It was proposed to figure out, for $n = 4$, which of a priori possible shapes of $W(B)$ are attained (for $n = 3$, this idea has been realized earlier). The participants showed experimentally, and then confirmed analytically, that all of them do. The next natural step (which, if successful, should lead to a publication) is to state the tests allowing to determine the shape type of $W(A)$ in terms of A directly. Three participants, most actively involved in this project, are planning to continue working in this direction (tentatively, with my participation).

2. Numerical ranges of 4×4 nilpotent matrices.

This is an off spring of a more general (and probably very hard) problem on a possible number of flat portions on the boundary of $W(A)$ for arbitrary $n \times n$ matrices A. For $n = 4$, this problem was solved completely by (a former William & Mary REU participant) E. Brown and myself, and the maximal number of such portions is then equal to four (three, if matrices are unitarily irreducible). However, for special matrices this number can be lower. It is easy to show that for nilpotent 4×4 matrices four is not attainable, and there is a conjecture (stated by Gau and Wu in their LAA'08 paper) that three is not attainable either.

One of the group members opted to work on this conjecture. Computer experiments did not generate a counterexample (which originally was our hope) but we are planning to continue working on this project and either to come up with such a counterexample, or (using the criteria established by Brown-Spitkovsky) to justify that indeed the maximal possible number of flat portions in the case under consideration is two.

Practical considerations for undergraduate research

This section is an attempt to summarize the discussion and represents opinions of some of the organizers and participants. UR = undergraduate research.

Models.

1-2 credit seminar.
Full credit research course.
Capstone project/senior thesis.
Independent study.
There are advantages and disadvantageous to year long vs. semester project.
Can even be multiyear.
Continuation contingent of faculty availability.
Consider interdisciplinary research.
UR can be a math major requirement, either there or at REU.
In addition to credit, can compensate students with money, such as research scholarship
Possible new models for an REU were discussed.

Prerequisites.

Often require courses through linear algebra, also a proof course.
Permission of instructor is a common requirement.

How often do you meet?.

Depends on model.
A course/seminar has a meeting time.
For independent study, once a week is common.

Selecting students/recruiting students.

It is fine to select students (e.g., limit to those who can benefit and/or limit number of students).
Depending on school culture and student finances you may have trouble recruiting.
Contests (COMAP, Putnam, etc.) can be recruiting tool (and sources of problems).
Colleges/universities push to have freshmen involved, but much math undergraduate research is not suitable for freshmen.

Group dynamics.

What if a student is not respectful of others? Talk to them individually.
What if one student dominates or one opts out? Assign individual tasks so everyone participates.

Funding.
Work with your Dean.
Industry.
NSF S-STEM provides scholarships for needy students if your department gets such a grant.

Funding opportunities from MAA (includes summer minority UR funding, conferences, travel grants),
http://www.maa.org/funding/undergraduate.html

MAA article on UR finding sources,
http://www.maa.org/columns/resources/resources_5_09.html

NSF REU-site and supplement grants,
http://www.nsf.gov/funding/pgm_summ.jsp?pims_id=5517

NSA grants, http://www.nsa.gov/research/math_research/index.shtml Center for Undergraduate Research in Mathematics (CURM) at BYU,
http://curm.byu.edu/ (not currently funded)

How do faculty get credit/find time?.

If your school has individual job descriptions, include UR in your job description– you should be evaluated on what it in writing.
Accumulate enough individual supervisions to equal a course.
UR meets expectation for advising or for teaching beyond course load.
Get buy-in from your chair and/or department.
Describe your request for release time as asking for "lab equipment"– time is the mathematics version of equipment.
Do not do too much before tenure (take care of yourself first).
small incentive payment (travel $)
Working with small groups rather than individually reduces the load.
You can choose the topic so amount you need to learn is reduced and may be possible to relate to your research.