Research experiences for undergraduate faculty

organized by
Leslie Hogben, Roselyn Williams, and Ulrica Wilson

Workshop Summary

This was the fourth in a series of workshops for undergraduate faculty. The goals of REUF are:

- To provide faculty participants the opportunity to have a research experience investigating open questions in the mathematical sciences, and enhance a love of doing original mathematics.
- To prepare participants to engage in research with undergraduate students at their home institutions, in some cases using the problems presented.
- To involve some participants in long-term research collaborations.
- To establish a network of faculty at primarily undergraduate institutions together with faculty at research universities who support collaboration and undergraduate research.

The four mathematical leaders, Nathaniel Dean of Texas State University, Stephan Garcia of Pomona College, Edray Goins of Purdue University, and Leslie Hogben of Iowa State University and AIM, presented problems Monday morning and research groups formed Monday afternoon. The groups worked throughout the week and presented reports Friday afternoon (see Sections 1 – 4. There were also whole group discussions of practical considerations of doing undergraduate research, the computer mathematics system *Sage*, and workshop continuation activities.

Minimum rank/Maximum nullity/Zero forcing

Leslie Hogben, Cheryl Grood, Johannes Harmse, Bonnie Jacob, Andrew Klimas, Sharon McCathern. Jillian McLeod started in this group but then moved on to investigate other problems, working with Stephan Garcia.

Report by Leslie Hogben

The standard minimum rank/maximum nullity problem involves real symmetric matrices described by a simple undirected graph. The *graph* $\mathcal{G}(A) = (V, E)$ of $n \times n$ symmetric matrix A is $G = (V, E)$ where $V = \{1, ..., n\}$ and $E = \{ij : a_{ij} \neq 0 \text{ and } i \neq j\}$ (the diagonal of A is ignored). The standard minimum rank/maximum nullity problem is to determine the maximum nullity $\mathrm{M}(G)$, or equivalently the minimum rank, of the matrices A having $\mathcal{G}(A) = G$.

Let G be a graph with each vertex colored either white or blue. The (standard) *color-change rule* is: if u is a blue vertex of G, and exactly one neighbor v of u is white, then change the color of v to blue. Given a coloring of G, the *final coloring* is the result of applying the color-change rule until no more changes result. A *zero forcing set* for a graph G is a subset of vertices B such that if initially the vertices in B are colored blue and the remaining vertices are colored white, the final coloring of G is all blue. The *zero forcing number* $\mathrm{Z}(G)$ is the minimum of $|B|$ over all zero forcing sets $B \subseteq V(G)$. It is known that $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

The group brainstormed a variety of questions related to variants of the standard maximum nullity, minimum rank and zero forcing number and decided to investigate maximum nullity, minimum rank and zero forcing number for symmetric matrices with zero diagonal. Numerous results have been obtained.

Prime and Coprime Labeling of Vertices and Edges

Nathaniel Dean, Adam H. Berliner, Jonelle Hook, Alison Marr, Aba Mbirika, Cayla McBee

Report by Nathaniel Dean

On the first day I gave a short talk entitled Prime and Coprime Labeling of Vertices and Edges to all REUF participants. I continued this introduction to my group covering subtopics and background material based on their interests.

A coprime labeling of a graph G of order n is a labeling of V with distinct positive integers so that the two labels on the ends of each edge are relatively prime. Every graph has a coprime labeling, and so the challenge is to find one that minimizes the size of the largest label. This number $\mathrm{pr}(G)$ is always at least n, and when $\mathrm{pr}(G)$ equals n, the graph is said to be *prime* (i.e., has a prime labeling). Roger Entringer (1980) conjectured that every tree is prime. This conjecture is still open; however, numerous families of graphs have been shown to be prime or not prime. We tried to prove Entringers conjecture and solve some other problems, and eventually we focused on trying to determine $\mathrm{pr}(G)$ for complete bipartite graphs and for ladders. A common method of research in mathematics is to compute examples, search for patterns, make conjectures, and then prove theorems. In fact we relied heavily on computing for every aspect of our research, except the proof. We established several new results, and by the end of the week everyone seemed pleased with what was accomplished mathematically and in terms of the new, tight network of collaborators that emerged and that will probably continue for the rest of our careers.

Although I explained that research with undergraduates doesnt have to center around solving open problems and that the students could benefit greatly by getting introduced to some new mathematical tools, it seemed that as untenured faculty they were very much concerned about their publication record and wanted to hear more about other types of open problems they could work on with students that would likely lead to a publication. So, I introduced them to a few other research areas in Graph Theory where a student would need only a minimal amount of mathematical preparation, some partial results could be obtained in a short time, and new results would likely be of interest to others in the field. Several of them plan to use these questions to encourage their students to talk about mathematics, formulate questions, and find solutions. If that happens (and I believe it will), I think the workshop went well.

Dessin d'Enfant

Edray Goins, Alejandra Alvarado, Naiomi Cameron, Emille Lawrence, Luis Melara, Karoline Null, Roselyn Williams

Report by Edray Goins

Main Project

Given a loopless, connected, planar, bipartite graph Γ, use properties of the symmetry group G to construct a Belyĭ map $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that Γ arises as its Dessin d'Enfant.

Summary of Results

- Every Belyĭ map $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree $\deg(f) = 1$ is in the form
$$f(z) = \frac{a\,z + b}{c\,z + d} \qquad \text{where} \qquad a\,d - b\,c \neq 0.$$
- Up to fractional linear transformation, every Belyĭ map $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree $\deg(f) = 2$ is in the form
$$f(z) = \left(\frac{a\,z + b}{c\,z + d}\right)^2 \qquad \text{where} \qquad a\,d - b\,c \neq 0.$$
- Consider four distinct complex numbers $z^{(-1)}$, $z^{(0)}$, $z^{(+1)}$, and $z^{(\infty)}$ with cross-ratio $\bigl(z^{(-1)},\, z^{(0)};\, z^{(+1)},\, z^{(\infty)}\bigr) = -1$ and define the rational function
$$f(z) = \left[\frac{2\left(z^{(0)} - z^{(1)}\right)\left(z^{(\infty)} - z^{(1)}\right)\left(z - z^{(0)}\right)\left(z - z^{(\infty)}\right)}{\left(z^{(0)} - z^{(1)}\right)^2\left(z - z^{(\infty)}\right)^2 + \left(z^{(\infty)} - z^{(1)}\right)^2\left(z - z^{(0)}\right)^2}\right]^2.$$
Then $f(z)$ is a Belyĭ map whose associated Dessin d'Enfant $K_{2,2}$ has vertices $B = \bigl\{z^{(0)},\, z^{(\infty)}\bigr\}$ and $W = \bigl\{z^{(-1)},\, z^{(+1)}\bigr\}$.
- Every planar complete bipartite graph $K_{m,n}$ be realized as the Dessin d'Enfant of some Belyĭ map, namely either $f(z) = z^n$ or $f(z) = 4\,z^n/(z^n + 1)^2$.
- Every path graph be realized as the Dessin d'Enfant of some Belyĭ map, namely $f(z) = \bigl(1 + \cos\left(n \arccos z\right)\bigr)/2$.
- Every bipartite cycle graph be realized as the Dessin d'Enfant of some Belyĭ map, namely $f(z) = (z^n + 1)^2/(4\,z^n)$.
- The Möbius Transformations $r(z) = (z - 1)/z$ and $s(z) = z/(z - 1)$ generate a subgroup of $\mathrm{Aut}\bigl(\mathbb{P}^1(\mathbb{C})\bigr)$ isomorphic to $S_3 = \bigl\langle r,\, s \,\big|\, r^3 = s^2 = (s\,r)^2 = 1\bigr\rangle$.
- Let $\phi(z)$ be a rational function. The composition $\phi \circ f$ is a Belyĭ map for every Belyĭ map f if and only if ϕ is a Belyĭ map which maps the set $\bigl\{(0:1),\, (1:1),\, (1:0)\bigr\}$ to itself.
- Let $\Gamma = \bigl(B \cup W,\, E\bigr)$ be the Dessin d'Enfant associated to a Belyĭ map $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. For each $\gamma(z) \in S_3$, let $\Gamma_\gamma = \bigl(B_\gamma \cup W_\gamma,\, E_\gamma\bigr)$ be the Dessin d'Enfant associated to the composition $\gamma^{-1} \circ f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. That is,
$$B_\gamma = f^{-1}\bigl(\gamma(0)\bigr), \qquad W_\gamma = f^{-1}\bigl(\gamma(1)\bigr), \qquad \text{and} \qquad E_\gamma = f^{-1}\bigl(\gamma([0,1])\bigr).$$
  - If $\gamma = 1$, then $\Gamma_1 = \Gamma$ is the original Dessin d'Enfant.
  - If $\gamma = s$, then Γ_s can be obtained from Γ by interchanging the white vertices W with the midpoints of the faces F.
  - If $\gamma = s\,r$, then Γ_s can be obtained from Γ by interchanging the black vertices B with the white vertices W. In other words, Γ_s is the dual graph to Γ.
  - If $\gamma = r\,s$, then Γ_s can be obtained from Γ by interchanging the black vertices B with the midpoints of the faces F.
  - If $\gamma = r$, then Γ_r can be obtained from Γ by cyclically rotating the black vertices B to the midpoints of the faces F to the white vertices W.

- If $\gamma = r^2$, then Γ_{r^2} can be obtained from Γ by cyclically rotating the black vertices B to the white vertices W to the midpoints of the faces F.

Problems related to Participant interests

Stephan Garcia, Ghan Batt, Sean Lawton, Bryant Mathews, Pedro Poitevin, Ellen Veomett

Report by Stephan Garcia

On Monday, our group started to work through some of the proposed problems and brainstorm about possible variations. We came to the conclusion that, given the diverse interests of the group (logic, functional analysis, graph theory, algebraic geometry,...) it might be better if we tried to come up with problems which were more suitable to the participants' own experiences and personal preferences. Throughout the rest of the week, each group member described their main research interests and previous experiences directing undergraduate research to the rest of the group. We would then brainstorm and come up with different undergraduate research projects which could be taken back to each member's home institution. For each of the group members, we were able to come up with 5-10 personalized undergraduate research questions which the respective participants felt capable and qualified to lead undergraduate research on. Throughout this process, we discussed different techniques and approaches to coming up with problems. On many occasions, I was also able to suggest relevant literature connected to these questions. In particular, it is hoped that the participants will later be able to tie their problems in to the existing literature, which would greatly increase chances of publication. All in all, I would like to think that the members of my group learned new ideas and approaches to creating suitable problems on their own. Toward the end, several of the group members were coming up with new and interesting questions on their own, and in their own intellectual territory (where they feel comfortable and capable of leading undergraduate research). I believe that the approach which we took was highly successful, and I hope that the feedback from the participants confirms this.