Riemann-Hilbert problems, Toeplitz matrices, and applications

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Workshop Summary

Report on the AIM workshop
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Introduction

Riemann-Hilbert problems (RHPs) and Toeplitz matrices play an important role in various areas of mathematics and applications to integrable probability, random matrix theory (RMT) and mathematical physics. Some of the most important examples include the 2D Ising model, quantum spin chain models, dimer models, and various results on the asymptotic behavior of large matrices, the study of random matrix models, the method of nonlinear steepest descent, and the study of integrable nonlinear PDEs. Another closely connected area includes special functions, such as Painlevé, which are important because they provide explicitly solvable models for a vast array of phenomena in mathematics and physics.

The workshop brought together leading experts and early-career researchers working in various areas connected to applications and theory of Riemann-Hilbert problems and concrete operators. Most of the work focused on the following problems, which are discussed in more detail below:

- Dimer models and Wiener-Hopf factorizations
- Planar orthogonal polynomials
- Riemann-Hilbert problems with jump matrices of size larger than two
- Double-scaling limits of Toeplitz determinants
- Berger-Coburn conjecture and Hilbert matrix operators

Dimer models and Wiener-Hopf factorizations

A certain Wiener–Hopf factorization problem appears naturally in the study of dimer models as the size of the underlying graph tends to infinity. In many cases, obtaining a Wiener-Hopf factorization of a matrix-valued function, denoted by ϕ, allows us to analyze this asymptotic behavior. The specific form of ϕ depends on the particular dimer model considered. A common setting involves a product form for ϕ, $\phi(z) = \prod_{i=1}^{N} \phi_i(z)$, where ϕ_i are explicit rational functions. These functions are typically such that the determinant of ϕ_i takes the form $(a - z^{\pm 1})^{\pm 1}$, for some $a \in \mathbb{R}$. While the Wiener-Hopf factorization can often be obtained explicitly for small values of N, the challenge lies in understanding its behavior as N approaches infinity.

For cases where $\phi_{i+k} = \phi_i$ for a fixed positive integer k and all i, the problem has been addressed by Berggren-Duits, Borodin-Duits, and Berggren-Borodin. The group focused on extending this analysis to a broader class of matrices ϕ beyond those previously considered. Specifically, we investigated matrices that arise naturally when dealing with a so-called q^{Volume}-measure with doubly periodic edge weights, more precisely,

$$\phi_i(z) = \begin{pmatrix} 1 & a_i q^i \\ b_i q^i & 1 \end{pmatrix} \tag{1}$$

where $a_{i+2} = a_i$ and $b_{i+2} = b_i$ for all i, and $q = \mathrm{e}^{\frac{1}{N}}$.

The group explored various methods for obtaining the Wiener-Hopf factorization for small values of N. We also discussed the possibility of an approximate solution using the fact that $q^i \sim \mathrm{e}^t$ for $t \in [0, 1]$ as i varies from 1 to N. While the discussions were fruitful, the problem remains challenging. Further research is needed to develop a robust approach to this specific case of Wiener-Hopf factorization.

Planar orthogonal polynomials

This working group started from questions and open problems proposed by H. Hedenmalm and K. McLaughlin during their talks at the AIM workshop "Riemann-Hilbert problems, Toeplitz matrices, and applications".

One of the goals is trying to match the ansatz proposed by Hedenmalm [Hedenmalm24] for analyzing planar Orthogonal Polynomials (OPs) with the $\bar{\partial}$-problem representation: in particular, we are trying to re-interpret the quantities appearing in Hedenmalm's ansatz in terms of the g-function, which drives the calculations in the $\bar{\partial}$-side.

Particular attention has been posed on the Hermite case as a "warm-up" model that can give insight and intuition on further generalization for other OPs: we are interested in understanding the intrinsic relation between the mother body (where the OPs zeros accumulate), the 2-dimensional droplet, the g-function, the 2D-potential Q, and the connection with 1-dimensional Hermite OPs. Within this setting, the matching will require a careful steepest descent analysis that involves the g-function and the Schwartz function describing the boundary of the droplet.

Finally, we are exploring the possibility of the existence of a map (that we call the $\mathcal{T}$ operator) between 1-dimensional (possibly on a curve Γ) and 2-dimensional OPs. In the Hermite case, the $\mathcal{T}$ operator maps Hermite polynomials into themselves (up to scaling constants). The questions are then 1) how does the $\mathcal{T}$ operator acts on different sets of OPs, 2) under what conditions the map is well defined and bijective, and 3) can the $\mathcal{T}$ operator be viewed as machine to generate (potentially new) OP systems.

Riemann-Hilbert problems with jump matrices of size larger than two

Riemann-Hilbert problems with jump matrices of size $k = 2$ are quite well understood and widely used in applications to random matrix theory and mathematical physics. The goal here was to discuss the problems with $k > 2$ and consider their applications. Such problems naturally arise in the context of random matrices with external source, which are ensembles of $n \times n$ Hermitian matrices equipped with the distribution

$$\frac{1}{Z_n} \exp\left\{-n\mathrm{Tr}\left(V(M) - AM\right)\right\} \mathrm{d}M, \tag{2}$$

where V is a polynomial and A is an $n \times n$ Hermitian matrix. Part of the activity in the group was to review some of the existing results—see [M-FS] and the references therein. As an example, a Riemann-Hilbert problem with a 4×4 jump matrix appears when in the ensemble (2) the Hermitian matrix A has three distinct eigenvalues a_1, a_2, and a_3 respectively with multiplicities n_1, n_2, and n_3 such that $n_1 + n_2 + n_3 = n$ which all grow with n. Participants discussed the prospects of answering the questions raised in [BK07] via analyzing the associated 4×4 Riemann-Hilbert problem. It is obvious that the larger size RHPs can be obtained in a similar way, but one needs to prioritize the analysis of the case $k = 4$.

Other areas where problems with $k > 2$ appear include phase diagram and topological expansion for the free energy for the quartic model with external source (see [BGM24]) and interpreting random matrix models with external source as special cases of two-matrix models. Besides the matrix models with external source, other large size Riemann- Hilbert problems appear in the study of radial toda equation of relatively small periodicity. Further, while not discussed during the workshop, the 4×4 problems also appear in connection with Toeplitz plus Hankel matrices.

Double-scaling limits of Toeplitz determinants

Double-scaling limits of Toeplitz determinants $D_n(f_t)$ as $n \to \infty$ and $t \to t_c$ (where the symbol depends on an additional parameter t) have received considerable attention in recent years because of various applications in random matrix theory and mathematical physics, in particular in the study of unitary invariant ensembles with certain potentials and the extreme values and averages of the characteristic polynomials of the circular unitary ensemble (such as Fyodorov-Keating conjectures [FK,FHK]), which are further related to the statistical properties of the Riemann zeta function. To complement recent results, which have all used the Riemann-Hilbert analysis, our discussions focused on operator-theoretic techniques and in particular on the use of insight gained in the recent PhD thesis [Pugh], which may lays the groundwork for further developments.

To be more specific, the discussion of the group focused on Toeplitz determinants $D_n(a)$ where the symbol is a piecewise continuous function $a = a_{t_1,\dots,t_R}$, representatable as

$$a_{t_1,\dots,t_R}(e^{ix}) = b(e^{ix}) \prod_{r=1}^{R} u_{\beta_r,t_r}(e^{ix}), \qquad u_{\beta,t}(z) := (-z/t)^{\beta}.$$

The case where the sizes β_r and the locations t_r of the jump discontinuites are fixed is described by the Fisher-Hartwig asymptotics [DIK1,DIK2]. But the case where the t_r's are not fixed but vary with n (while the β_r's are still fixed), has not yet been systematically been studied (see however [CK,CF]). In view of the dependence of $t_1, \dots, t_R$ on n a whole variety of scenarios is possible, which may lead to different types of asymptotic behavior. The group focused on two of them as a starting point.

(1) Assume that, as $n \to \infty$ we have $t_r \to t_0$ (i.e., the location of the jumps approaches a single (fixed) point t_0). Under what conditions does the ratio

$$\frac{D_n(a_{t_1,\dots,t_R})}{D_n(a_{t_0,\dots,t_0})}$$

converge to a constant? (The asymptotics of the denominator is described by classical Fisher-Hartwig.) Based on insight gained from corresponding stability analysis [Pugh], a case where progress seems possible is when the (multiple) scaling limits

$$n(t_r/t_0 - 1) \to ix_r$$

with $x_r \in \mathbb{R}$ fixed hold. In other words, when the t_r's approach the t_0 with some "moderate" speed, $t_r \approx t_0 e^{ix_r/n}$. The details need to be worked out and may require some effort.

(2) Another interesting case is when the symbol is a product of two functions, both of the kind as described in (1), but with different t_0's. Say, we have symbols

$$a_{t_1,\dots,t_R} \text{ with } t_r \to t_0, \qquad 1 \le r \le R$$

and

$$a_{\tilde{t}_1,\dots,\tilde{t}_S} \text{ with } \tilde{t}_s \to \tilde{t}_0, \qquad 1 \le s \le S,$$

$t_0 \neq \tilde{t}_0$. Then it seems plausible that "classical" localization results could be proved, i.e., the limit

$$\frac{D_n(a_{t_1,\dots,t_R} a_{\tilde{t}_1,\dots,\tilde{t}_S})}{D_n(a_{t_1,\dots,t_R}) D_n(a_{\tilde{t}_1,\dots,\tilde{t}_S})}$$

converges to a constant. A proof of such a result would still be non-trivial since it will rely on (double/multiple) scaling limit versions of stability of sequences of Toeplitz matrices in the spirit of [Pugh].

In the above discussion, the sizes β_r are supposed to be sufficiently small, i.e., we are not attempting to deal with a generalized FH-asymptotics.

Berger-Coburn conjecture and Hilbert matrix operators

Berger and Coburn [berger1994heat] show that for a Toeplitz operator $T_g : F \to F$ acting on the Segal–Bargmann space F of all entire functions that are square-integrable with respect to the Gaussian measure one has the estimates

$$C(t) \|T_g\| \ge \left\|\tilde{g}^{(t)}\right\|_\infty, \quad 1 > t > \frac{1}{4}$$

$$C(t) \left\|\tilde{g}^{(t)}\right\|_\infty \ge \|T_g\|, \quad 0 < t < \frac{1}{4},$$

where

$$\tilde{g}^{(t)}(a) = \int_{\mathbb{C}^n} g(w) \exp\left\{-|w-a|^2/4t\right\} dv(w)(4\pi t)^{-n}.$$

This leads to a natural conjecture (now known as the Berger-Coburn conjecture) that T_g is bounded if and only if $\tilde{g}^{(1/4)}$ is bounded.

The focus of our work group was twofold. We first exploited the (unitary) Bargmann transform $B : L^2(\mathbb{R}) \to F$, and wrote down an integral representation for the map $B^*FB : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Clearly F is bounded if and only B^*FB is bounded. So we tried to see how one could algebraically (or otherwise) manipulate the integral expression B^*FB so that $\tilde{g}^{(t)}(a)$ appears.

Our second approach was to find a norm attaining sequence of vectors for the Toeplitz operator T_g. Since the Segal–Bargmann space is a reproducing kernel Hilbert space in which the reproducing kernels $k_a(z)$ satisfy

$$\langle T_g k_a, k_a \rangle = \tilde{g}^{(1/2)}(a)$$

it seemed plausible that if $x_n = \sum x_i^n k_{a_i}^n$ is a norm attaining sequence for T_g, then one may be able to express $\|T_g x_n\|$ in terms of $\tilde{g}^{(1/4)}$.

Since the original proofs of $C(t)\left\|T_g\right\| \geq \left\|\tilde{g}^{(t)}\right\|_\infty$, for $1 > t > \frac{1}{4}$ and $C(t)\left\|\tilde{g}^{(t)}\right\|_\infty \geq \left\|T_g\right\|$, for $0 < t < \frac{1}{4}$ use trace norm duality results, our hope was if one could examine and reproduce these results using constructive methods, this would shed light on the Berger-Coburn Conjecture.

In addition to the Berger-Coburn conjecture, we also discussed the Hilbert matrix operators $\mathcal{H}_\lambda$, which in the simplest form can be understood as the matrices

$$\mathcal{H}_\lambda = \left(\frac{1}{j+k+\lambda}\right)_{j,k\geq 0}$$

acting on ℓ^2, where $\lambda \neq 0, -1, -2, \ldots$. In particular, we reviewed some of the previous proofs of Magnus and Hill of the results on the spectral properties of $\mathcal{H}_1$, which are quite complicated and involved, and provided in [M-RV] a more streamlined and simple proofs of their results using the properties of the Mehler-Fock transform. We also considered the Hilbert matrix as an operator acting on the Bergman spaces A^2_λ when $-1 < \lambda < 0$ and tried to compute its norm. We largely reviewed the relevant previous work on this difficult, long-standing problem, and made plans for further work that arose from our discussions on generalized Mehler-Fock transforms.

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